Received: *November 20, 1998*, Accepted: *December 21, 1998*

D-branes and K-theory

Edward Witten

School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA
E-mail: witten@ias.edu

Abstract: By exploiting recent arguments about stable nonsupersymmetric D-brane states, we argue that D-brane charge takes values in the K-theory of space-time, as has been suspected before. In the process, we gain a new understanding of some novel objects proposed recently – such as the Type I zerobrane – and we describe some new objects – such as a -1-brane in Type I superstring theory.

Keywords: D-branes, Superstring Vacua, String Duality.

JHEP12(1998)019

Contents

1. Introduction **1**

2. Some questions about Type I superstrings **3**

3. Brane-antibrane annihilation and K-theory **7**

4. Incorporating p-branes with $p < 9$ **11**
- 4.1 Review of Sen's construction 11
- 4.2 Global version 13
- 4.3 Spinors and the anomaly 15
- 4.4 Spinors and Type I branes 18
- 4.5 The Type I zerobrane and −1-brane 21

5. Some generalizations **25**
- 5.1 Orbifolds 25
- 5.2 Involutions 26
- 5.3 Incorporation of the B-field 27

6. Stringy constructions for Type I **30**
- 6.1 The zerobrane 31
- 6.2 The −1-brane 35

1. Introduction

One of the most important insights about nonperturbative behavior of string theory is that D-branes carry Ramond-Ramond charge [1]. Massless Ramond-Ramond fields are differential forms, and therefore the Ramond-Ramond charges would appear to be cohomology classes — measured by integrating the differential forms over suitable cycles in the spacetime manifold X.

There have, however, also been reasons to suspect that one should understand D-brane charges in terms of the K-theory of spacetime rather than cohomology. First of all, gauge fields propagate on D-brane worldvolumes; this is more suggestive of K-theory — which involves vector bundles and gauge fields — than of cohomology. If, moreover, a D-brane wraps on a submanifold Y of spacetime, then its Ramond-Ramond charges depend on the geometry of Y and of its normal bundle, and on the gauge fields on Y, in a way that is suggestive of K-theory [2, 3]. Furthermore, the treatment of D-branes on an orbifold [4] is reminiscent of equivariant K-theory. Finally, one can see Bott periodicity in the brane spectrum of Type IIB, Type IIA, and Type I superstrings. (For Type II, one has unitary gauge groups in every even or every odd dimension, and for Type I, one flips from SO to Sp and back to SO in adding four to the brane dimension; these facts are reminiscent of the periodicity formulas $\pi_i(U(N)) = \pi_{i+2}(U(N))$ and $\pi_i(SO(N)) = \pi_{i\pm 4}(Sp(N))$.) Such arguments motivated a proposal [3] that D-brane charge takes value in $\mathrm{K}(X)$. (The proposal was accompanied by a remark that the torsion in $\mathrm{KO}(X)$ would provide simple and interesting examples.) Also, a possible relation of orientifolds with KR-theory was briefly mentioned in [5].

In another line of development, stable but nonsupersymmetric (that is, non-BPS) states in string theory have been investigated recently [6, 7]. It has been shown that in many instances these are naturally understood as bound states of a brane-antibrane system with tachyon condensation [8, 9, 10], and more concretely as novel stable but nonsupersymmetric D-branes [11, 7]. Brane anti-brane annihilation in the special case of ninebranes — which will be important in the present paper — has been discussed in [12].

The main purpose of the present paper is to bring these two lines of development together, by showing that the methods that have been used in analyzing the brane-antibrane system lead naturally to the identification of D-brane charge as an element of $\mathrm{K}(X)$ — the K-theory of the spacetime manifold X. In the process, we will gain some new understanding of constructions that have been made already, and will propose some new constructions.

The paper is organized as follows. Section 2 is offered as an appetizer – some simple questions about Type I superstring theory are posed, and intuitive answers are given that we will seek to understand better through the rest of the paper. The basic relation of the brane-antibrane system to K-theory is explained in section 3. The identification of D-brane charges with K-theory is completed in section 4. The main idea here is that Sen's description of brane-antibrane bound states (as presented most fully in [10]) can be identified with a standard construction in K-theory, involving the Thom isomorphism or Bott class. In concluding the argument, one also needs a topological condition that has been noticed previously [13] and can be understood as a worldsheet global anomaly [14] but which has hitherto seemed rather obscure. In section 5, we generalize the discussion to orbifolds and orientifolds, and also to include the Neveu-Schwarz three-form field H, which is assumed to vanish in most of the paper. In section 6, we discuss worldsheet constructions for some interesting special cases, including a Type I zerobrane that has been discussed before, and a new Type I -1-brane.

For more background on K-theory and fuller explanations of some constructions that we will meet later, the reader might consult [15, 16]. I have generally tried to make this paper self-contained and readable with no prior familiarity with K-theory, though certain assertions will be made without proof.

2. Some questions about Type I superstrings

We begin by asking some questions about Type I superstring theory on $\mathbf{R}^{10}$, and proposing intuitive answers; we will reexamine these questions in sections 4 and 6.

The gauge group of the Type I superstring is locally isomorphic to $SO(32)$. The global form of the group is not precisely $SO(32)$ and will be discussed later. We will also compare later with the perturbative $SO(32)$ heterotic string. Our interest will focus on some of the homotopy groups of $SO(32)$, namely

$$\begin{aligned}\pi_7(SO(32)) &= \mathbf{Z}\,,\\ \pi_8(SO(32)) &= \mathbf{Z}_2\,,\\ \pi_9(SO(32)) &= \mathbf{Z}_2\,.\end{aligned} \tag{2.1}$$

$SO(32)$ bundles on the $i{+}1$-dimensional sphere $\mathbf{S}^{i+1}$ are classified by $\pi_i(SO(32))$. The following relations of the homotopy groups just introduced to index theory will be important presently: the topological charge of an $SO(32)$ bundle on $\mathbf{S}^8$ is measured by the index of the Dirac operator; a nontrivial $SO(32)$ bundle on $\mathbf{S}^9$ is characterized by having an odd number of zero modes of the Dirac operator; a nontrivial bundle on $\mathbf{S}^{10}$ is similarly characterized by having an odd number of zero modes of the *chiral* Dirac operator.

$SO(32)$ bundles on $\mathbf{S}^{i+1}$ are equivalent to $SO(32)$ bundles on Euclidean space $\mathbf{R}^{i+1}$ that are trivialized at infinity (the trivialization means physically that the gauge field is pure gauge at infinity and the action integral on $\mathbf{R}^{i+1}$ converges). So, in ten-dimensional spacetime, we can seemingly use π_7, π_8, and π_9 to construct strings, particles, and instantons, respectively. What are these objects?

This question is outside the reach of low energy effective field theory for the following reason. Non-zero $\pi_i(SO(32))$ for $i = 7, 8, 9$ leads to the existence of topologically non-trivial gauge fields on $\mathbf{R}^{i+1}$, but those objects do not obey the Yang-Mills field equations. A simple scaling argument shows that for $n > 4$, the action of any gauge field on $\mathbf{R}^n$ (defined in low energy effective field theory as $\frac{1}{4}\int d^nx\ \mathrm{Tr}F_{ij}F^{ij}$) can be reduced by shrinking it to smaller size. So the objects associated with π_7, π_8, and π_9, though they can be constructed topologically using long wavelength gauge fields, will shrink dynamically to a stringy scale.

Nevertheless, by using low energy effective field theory, we can guess intuitively the interpretation of these objects:

The string. The string associated with $\pi_7(SO(32))$ — let us call it the gauge string — can be identified as follows.[1] Let B be the two-form field of Type I superstring theory. It is a Ramond-Ramond field, and couples to the D-string. However, B

[1]This object has actually been first constructed in [17], where the coupling to the B-field was computed.

also couples to the gauge string because of the Green-Schwarz anomaly canceling term $\int B \wedge (\mathrm{Tr}\, F^4 + \ldots)$, since the gauge string is made from a gauge field on $\mathbf{R}^8$ with a nonzero integral $\int_{\mathbf{R}^8} (\mathrm{Tr}\, F^4 + \ldots)$. In fact, as we will presently calculate, the minimal gauge string has D-string charge ± 1. This strongly suggests that the string constructed in low energy field theory using a generator of $\pi_7(SO(32))$ shrinks dynamically to an ordinary D-string.

To compute the D-string charge of the gauge string, let V be an $SO(32)$ bundle on $\mathbf{R}^8$ with a connection of finite action. Because the connection is flat at infinity, we can compactify and regard V as an $SO(32)$ bundle on $\mathbf{S}^8$. This bundle has $p_1(V) = 0$ (since $p_1(V)$ would take values in $H^4(\mathbf{S}^8)$, which vanishes), and

$$\int_{\mathbf{S}^8} p_2(V) = 6k\,, \tag{2.2}$$

where k is an arbitrary integer. The factor of 6 arises as follows. As we remarked above, the topological charge of an $SO(32)$ bundle V on $\mathbf{S}^8$ is measured by the Dirac index, which can be — depending on the choice of V — an arbitrary integer k. On the other hand, using the index theorem, the Dirac index for spinors on $\mathbf{S}^8$ valued in V is

$$\int_{\mathbf{S}^8} \mathrm{ch}(V) = \sum_i \int_{\mathbf{S}^8} \left(e^{\lambda_i} + e^{-\lambda_i}\right) = -\int_{\mathbf{S}^8} \frac{p_2(V)}{6}\,. \tag{2.3}$$

Here λ_i are the roots of the Chern polynomial, the Pontryagin classes are $p_1 = \sum_i \lambda_i^2$ (which vanishes) and $p_2 = \sum_{i<j} \lambda_i^2 \lambda_j^2$, and ch is the Chern character. So $p_2(V)$ can be any multiple of 6.

On the other hand, the standard anomaly twelve-form (the one-loop anomaly of the massless gravitinos and gluinos of the Type I theory) is

$$-\frac{1}{2}\left(p_1(V) - p_1(T)\right) \cdot \left(\frac{p_2(V)}{6} + \ldots\right), \tag{2.4}$$

where the $\ldots$ are terms not involving $p_2(V)$. Since the field strength H of the B-field (normalized so that the periods of B are multiples of 2π) obeys $dH = \frac{1}{2}(p_1(V) - p_1(T))$, the properly normalized coupling of B to $p_2(V)$ is

$$\int B \wedge \frac{p_2(V)}{6}\,. \tag{2.5}$$

Since $p_2(V)/6$ can be any integer, it follows that the minimal gauge string has D-string charge 1.

The particle. We now consider the particle associated with $\pi_8(SO(32))$; let us call it the gauge soliton. We claim that — in contrast to elementary Type I string states, which transform as tensors of $SO(32)$ — the gauge soliton transforms in a spinorial representation of $SO(32)$ (a representation of Spin(32) in which a 2π rotation acts by -1). It can thus, potentially, be compared to the D-particle found in [10, 7], which transforms in this way.

To justify the claim, we argue as follows. The gauge soliton is described by a nontrivial $SO(32)$ bundle V on $\mathbf{R}^9$, or — after compactification — on $\mathbf{S}^9$. One can pick a connection on V that lives in an $SO(n)$ subgroup of $SO(32)$, for any n with $n \geq 9$. Such a connection leaves an unbroken subgroup $H = SO(32-n)$. We will argue that the gauge soliton transforms in a spinorial representation of $SO(32)$ by showing that it is odd under a 2π rotation in H. We write $V = U \oplus W$, with U a non-trivial $SO(n)$ bundle and W a trivial H bundle.

As in many such problems involving charge fractionation [18], the essence of the matter is to look at the zero modes of the Dirac operator. In Type I superstring theory, the massless fermions that are not neutral under $SO(32)$ are gluinos, which transform in the adjoint representation. The gluinos that transform non-trivially under H and also "see" the $SO(n)$ gauge fields transform as $(\mathbf{n}, \mathbf{32}-\mathbf{n})$ of $SO(n)\times H$ and are sections of $U \otimes W$. The Dirac operator with values in U has an odd number of zero modes; generically, this number is one. So under H, one has generically a single vector of fermion zero modes. Its quantization gives states transforming in the spinor representation of H, supporting the claim that the gauge soliton is odd under a 2π rotation in H and hence transforms in a spinorial representation of $SO(32)$.

This supports the idea that the nonperturbative gauge group of the Type I superstring is really a spin cover of $SO(32)$. Duality with the heterotic string indicates that the gauge group is really $\mathrm{Spin}(32)/\mathbf{Z}_2$ (rather than $\mathrm{Spin}(32)$). Possibly this could be seen in the present discussion by quantizing the bosonic collective coordinates of the gauge instanton (which break $SO(32)$ down to H). We will not attempt to do so.

The instanton. The perturbative symmetry group of the Type I superstring is actually more nearly $O(32)$ than $SO(32)$, as orthogonal transformations of determinant -1 are symmetries of the perturbative theory. (To be more precise, the central element -1 of $O(32)$ acts trivially in Type I perturbation theory, so the symmetry group in perturbation theory is $O(32)/\mathbf{Z}_2$.) Duality with the heterotic string indicates that the transformations of determinant -1 are actually not symmetries, so we must look for a nonperturbative effect that breaks $O(32)$ to $SO(32)$. I will now argue that the instanton associated with $\pi_9(SO(32))$ —- call it the gauge instanton (in the present discussion there should be no confusion with standard Yang-Mills instantons!) — has this effect.

The analysis is rather like what we have just seen. The ten-dimensional gauge instanton can be deformed to lie in a subgroup $SO(n)$ of $O(32)$ (with any $n \geq 10$), and so to leave an unbroken subgroup $H = O(32-n)$. Again we decompose the $O(32)$ bundle as $V = U \oplus W$, with U a non-trivial $SO(n)$ bundle and W a trivial H bundle. To test for invariance under the disconnected component of $O(32)$, we let w be an element of the disconnected component of H and ask whether w leaves the quantum measure in the instanton field invariant. As usual, this amounts to asking whether the measure for the fermion zero modes is invariant under w — since everything else is invariant. The fermions that are not neutral under $O(32)$ are the gluinos. As in the discussion of the gauge soliton, the relevant gluinos transform as $(\mathbf{n}, \mathbf{32}-\mathbf{n})$ under $SO(n) \times H$ and are sections of $U \otimes W$. The Dirac equation for (Majorana-Weyl) fermions with values in U has an odd number of fermion zero modes — generically one. So the fermion zero modes that are not H-invariant consist of an odd number of vectors of H. The measure for the zero modes is therefore odd under the disconnected component of H, supporting the claim that the gauge instanton breaks the invariance under the disconnected component of $O(32)$.

The existence of an instanton for Type I seems at first sight to mean that this theory has a discrete theta angle: one could weight the instanton amplitude with a $+$ sign or a $-$ sign. However, the two choices give equivalent theories since a transformation in the disconnected component of $O(32)$ changes the sign of the instanton amplitude.

Comparison to the heterotic string. Now let us consider how we might interpret the gauge string, soliton, and instanton for the $\mathrm{Spin}(32)/\mathbf{Z}_2$ heterotic string.

All three objects are manifest in heterotic string perturbation theory. We have interpreted the gauge string as the Type I D-string, which corresponds to the perturbative heterotic string; the gauge soliton as a particle in the spinor representation, like some of the particles in the elementary heterotic string spectrum; and the gauge instanton as a mechanism that breaks the disconnected component of $O(32)$ — a breaking that is manifest in heterotic string perturbation theory.

So from the point of view of the heterotic string, it seems that the ostensibly non-perturbative gauge string, soliton, and instanton can all be continuously converted to ordinary perturbative objects. But they are relevant to understanding weakly coupled Type I superstring theory.

Relation to the rest of this paper. In all the above, it was not material that the Type I gauge group is precisely $SO(32)$. Any orthogonal gauge group of large enough rank would have served just as well (one has $\pi_i(SO(k)) = \pi_i(SO(k+1))$ if $k > i$). It would have been more convenient if we could have somehow enlarged the gauge group from $SO(32)$ to $SO(32+n)$ for some $n > 9$. Then we could have carried out the above arguments with a manifest $SO(32)$ symmetry, instead of seeing only a subgroup H. The constructions given recently in [8, 9, 10] permit one to make the discussion with an enlarged gauge group. This enlargement is related to K-theory. In sections 4 and 6, after learning more, we will reexamine from new points of view the topological defects that we have discussed above.

JHEP12(1998)019

3. Brane-antibrane annihilation and K-theory

Consider in Type II superstring theory a p-brane and an anti p-brane both wrapped on the same submanifold W of a spacetime X. We will use the term $\overline{p}$-brane as an abbreviation for anti p-brane. Intuitively, one would expect that as there is no conserved charge in the system of coincident p-brane and $\overline{p}$-brane, they should be able to annihilate.

This is supported as follows by the analysis of the brane-antibrane pair. Low-lying excitations of this system are described, as usual, by p-p, p-$\overline{p}$, and $\overline{p}$-$\overline{p}$ open strings. The p-p open string spectrum consists of a massless super Maxwell multiplet plus massive excitations. The familiar NS sector tachyon is removed by the GSO projection. The $\overline{p}$-$\overline{p}$ open strings give another super Maxwell multiplet. However, for the p-$\overline{p}$ and $\overline{p}$-p open strings, one must make the opposite GSO projection. Hence, the massless vector multiplet is projected out, and the tachyon survives [19, 20]. It is conjectured that the instability associated with the tachyon represents a flow toward annihilation of the brane-antibrane pair. In other words, by giving the tachyon field a suitable expectation value, one would return to the vacuum state without this pair.[2] This has been argued [9] using techniques in [21]. Brane-antibrane annihilation can also be seen semiclassically [22, 23].

The fact that the p-$\overline{p}$ and $\overline{p}$-p strings have a reversed GSO projection can be formalized as follows. Consider the p-$\overline{p}$ brane system to have a two-valued Chan-Paton label i, where $i = 1$ for an open string ending on the p-brane and $i = 2$ for an open string ending on the $\overline{p}$-brane. Thus at the end of the string lives a charge that takes values in a two-dimensional quantum Hilbert space. Consider the $i = 1$ state to be bosonic and the $i = 2$ state to be fermionic. Thus, the GSO projection operator $(-1)^F$, which usually acts trivially on the Chan-Paton factors, acts here by

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

The p-p and $\overline{p}$-$\overline{p}$ open strings have diagonal Chan-Paton wave functions. These wave functions are even under $(-1)^F$, leading to the usual GSO projection on the oscillator modes. The Chan-Paton wavefunctions for p-$\overline{p}$ and $\overline{p}$-p open strings are off-diagonal and odd under $(-1)^F$, leading to a reversed GSO projection for the oscillators. (Note that it would not matter if we multiply the right hand side of (3.1) by an overall factor of -1; in the action of $(-1)^F$ on string states, this factor will cancel out, as each open string has two ends.) Having one bosonic and one fermionic

[2] There is a puzzle about this process even at a heuristic level [12]. The gauge group of the brane-antibrane pair is $U(1) \times U(1)$, with one $U(1)$ on the brane and one on the antibrane. The tachyon field T has charges $(1, -1)$, and its expectation value breaks $U(1) \times U(1)$ to a diagonal $U(1)$ subgroup. This $U(1)$ must ultimately be eliminated in the brane-antibrane annihilation, but it is not clear how this should be described.

JHEP12(1998)019

Chan-Paton state would lead, if we made no GSO projection, to a gauge supergroup $U(1|1)$. Because of the GSO projection, the off-diagonal fermionic gauge fields of $U(1|1)$ are absent, and we get instead a structure whose lowest modes correspond to a "superconnection" (in the language of [24]), that is to a matrix of the form

$$\begin{pmatrix} A & T \\ \overline{T} & A' \end{pmatrix}, \tag{3.2}$$

where A and A' are the gauge fields and T is the p-$\overline{p}$ tachyon. If A and A' are connections on bundles E and F (E and F are the bundles of "bosonic" and "fermionic" Chan-Paton states), then T is a section of $E \otimes F^*$ and $\overline{T}$ of $E^* \otimes F$. (E^* denotes the dual of a bundle E.) In section 6, we will encounter more exotic actions of $(-1)^F$ on the Chan-Paton wavefunctions.

Now, let us consider a more general case with n p-branes and n $\overline{p}$-branes wrapped on the submanifold W of spacetime. We allow an arbitrary $U(n)$ gauge bundle E for the p-branes, and (topologically) the *same* bundle for the $\overline{p}$-branes. The reason for picking the same gauge bundle for both branes and antibranes is to ensure that the overall system carries no D-brane charges. (The operator $(-1)^{F_L}$ maps p-branes to $\overline{p}$-branes, and reverses the sign of all D-brane charges, while leaving fixed the gauge fields on the branes.) Since this system carries no evident conserved charges, and there is certainly a tachyon in the p-$\overline{p}$ sector, one would expect that any such collection of branes can annihilate. This is the basic technical assumption that we will make in what follows.

Now to proceed, we will specialize first to the case of Type IIB superstrings, and we will consider first the important special case of what can be achieved using only 9-branes and $\overline{9}$-branes. p-branes with $p < 9$ will be included in the next section.

We start with an arbitrary configuration with n 9-branes and the same number of $\overline{9}$-branes. (Tadpole cancellation is the only reason to require the same number of 9-branes and $\overline{9}$-branes.) In general, the 9-branes carry a $U(n)$ gauge bundle E, and the $\overline{9}$-branes carry a $U(n)$ gauge bundle F. We label this configuration by the pair (E, F).

Now, we ask to what other configurations (E', F') the configuration (E, F) is equivalent. The basic equivalence relation we assume is brane-antibrane creation and annihilation, as described above. We suppose that any collection of m 9-branes and m $\overline{9}$-branes, with the same $U(m)$ gauge bundle H for both branes and antibranes, can be created or annihilated. So the pair (E, F) can be smoothly deformed to $(E \oplus H, F \oplus H)$. Since we are only interested in keeping track of conserved D-brane charges — properties that are invariant under smooth deformations — we consider the pair (E, F) to be equivalent to $(E \oplus H, F \oplus H)$.

What we have just arrived at is the definition of the K-group $\mathrm{K}(X)$. $\mathrm{K}(X)$ is defined by saying that an element of $\mathrm{K}(X)$ is a pair of complex vector bundles (E, F) over spacetime, subject to an equivalence relation which is generated by saying that

(E, F) is equivalent to $(E \oplus H, F \oplus H)$ for any H. $\mathrm{K}(X)$ is a group, the sum of (E, F) and (E', F') being $(E \oplus E', F \oplus F')$.[3] One sometimes writes (E, F) as $E - F$. The subgroup of $\mathrm{K}(X)$ consisting of elements such that E and F have the same rank (equal numbers of 9-branes and $\overline{9}$-branes) is usually called $\widetilde{\mathrm{K}}(X)$. Thus, we conclude that tadpole-cancelling 9-$\overline{9}$ configurations, modulo creation and annihilation of brane pairs, are classified by $\widetilde{\mathrm{K}}(X)$.

At this point, we can explain why and to what extent it is a good approximation to think of D-brane charge as taking values in cohomology rather than K-theory. For a vector bundle E, let $c(E) = 1 + c_1(E) + c_2(E) + \ldots$ denote the total Chern class. The total Chern class of a K-theory class $x = (E, F)$ is defined as $c(x) = c(E)/c(F)$, the point being that this is invariant under $(E, F) \to (E \oplus G, F \oplus G)$. ($1/c(F) = 1/(1 + c_1(F) + c_2(F) + \ldots)$ is defined by expanding it in a power series as $1 - c_1(F) + c_1(F)^2 - c_2(F) + \ldots$.) The component of $c(x)$ of dimension $2k$ is written $c_k(x)$ and called the k^{th} Chern class of x. Measuring D-brane charge by cohomology rather than K-theory amounts to measuring a K-theory class by its Chern classes $c_k(x)$. This gives a somewhat imprecise description as there are K-theory classes whose Chern classes are zero, and is also awkward because there is no natural description purely in terms of cohomology of precisely which sequences of cohomology classes arise as Chern classes of some $x \in \mathrm{K}(X)$. However, using cohomology instead of K-theory is an adequate approximation if one is willing to ignore multiplicative conservation laws (associated with torsion classes in K-theory) and one in addition does not care about the precise integrality conditions for D-brane charge.

We conclude this section with some technical remarks. The spacetime X is usually noncompact, for instance $X = \mathbf{R}^4 \times Q$ where Q may be compact. Because of a finite action or finite energy restriction, one usually wants objects that are equivalent to the vacuum at infinity. Here, "equivalent to the vacuum at infinity" means that near infinity, one can relax to the vacuum by tachyon condensation. In many cases, there are no branes in the vacuum, in which case "equivalent to the vacuum at infinity" means that in the pair (E, F), E is isomorphic to F near infinity. In general the vacuum may contain branes and thus may be represented by a nonzero K-theory class.[4] "Infinity" means spacetime infinity if one is considering

[3] It is actually a ring, with the product of (E, F) and (E', F') being $(E \otimes E' \oplus F \otimes F', E \otimes F' \oplus F \otimes E')$, as if the E's are bosonic and the F's fermionic. This multiplication will not be exploited in the present paper.

[4] Tadpole cancellation, or in other words the condition that the equations of motion of Ramond-Ramond fields should have solutions, typically determines the K-theory class of the vacuum in terms of geometric data. For instance, jumping ahead of our story a bit, for Type I superstrings, the ninebrane charge is 32 for tadpole cancellation, the fivebrane charge is determined by the equation $dH = \frac{1}{2}(\mathrm{Tr}F^2 - \mathrm{Tr}R^2)$, and (if we compactify to two dimensions) the onebrane charge is determined by the fact that the integrated source of the B-field, appearing in the Green-Schwarz coupling $\int B \wedge \left(\mathrm{Tr}F^4 + \ldots\right)$, must vanish.

JHEP12(1998)019

instantons, spatial infinity in the case of particles, infinity in the normal directions for strings, and so on. Requiring that the class (E, F) is equivalent to the vacuum at infinity means that if we subtract from (E, F) the K-theory class of the vacuum, we get a K-theory class (E', F') that is trivial at infinity (in the sense that E' and F' are isomorphic at infinity). The Ramond-Ramond charge of an excitation of a given vacuum is best measured by subtracting from its K-theory class the K-theory class of the vacuum.

Hence in most physical applications, the Ramond-Ramond charge of an excitation of the vacuum is most usefully considered to take values not in the ordinary K-group $\mathrm{K}(X)$, but in K-theory with compact support. More precisely, for instantons one uses K-theory with compact support, for particles one uses K-theory with compact support in the spatial directions, etc. A K-theory class with compact support is always represented by a pair of bundles of equal rank, since bundles that are isomorphic at infinity must have equal rank. So the distinction between $\mathrm{K}(X)$ and $\widetilde{\mathrm{K}}(X)$ is inessential for most physical applications. Hence, we will describe our result somewhat loosely by saying that, up to deformation, 9-brane excitations of a Type IIB spacetime X are classified by $\mathrm{K}(X)$, with the understanding that the precise version of $\mathrm{K}(X)$ which is relevant depends on the particular situation that one considers.

Other string theories. Now we consider other theories with D-branes, namely Type I and Type IIA.

For Type I, the discussion carries over with a few simple changes. We consider a system with n 9-branes and m $\overline{9}$-branes. Tadpole cancellation now says that $n - m = 32$. The branes support an $SO(n)$ bundle E and an $SO(m)$ bundle F. By brane-antibrane creation and annihilation, we assume that the pair (E, F) is equivalent to $(E \oplus H, F \oplus H)$ for any $SO(k)$ bundle H.

Pairs E, F with this equivalence relation (and disregarding for the moment the condition $n - m = 32$), define the real K-group of spacetime, written $\mathrm{KO}(X)$. The subgroup with $n - m = 0$ is called $\widetilde{\mathrm{KO}}(X)$. Any configuration with $n - m = 32$ can be naturally mapped to $\widetilde{\mathrm{KO}}(X)$ by adding to F a rank 32 trivial bundle. So pairs (E, F) subject to the equivalence relation and with $n - m = 32$ are classified by $\widetilde{\mathrm{KO}}(X)$. As we noted in discussing Type IIB, in most physical applications, one wishes to measure the K-theory class of an excitation relative to that of the vacuum. If we do so, then the brane charge of an excitation is measured by $\widetilde{\mathrm{KO}}(X)$ with a compact support condition. With such a compact support condition KO and $\widetilde{\mathrm{KO}}$ are equivalent, so we will describe our result by saying that 9-$\overline{9}$ configurations of Type I are classified by $\mathrm{KO}(X)$.[5]

[5]Unlike what we said for Type IIB, this identification of 9-brane configurations for Type I with KO-theory does not really require assumptions about brane-antibrane annihilation, in the following sense. Since X has dimension 10, the classification of $SO(32)$ bundles on X is governed by the

The discussion for Type IIA is more subtle, because the brane world-volumes have odd codimension. I will not attempt a complete description in the present paper. The basic idea is to relate branes not to bundles on X but to bundles on $\mathbf{S}^1 \times X$.[6] Given a p-brane wrapped on an odd-dimensional submanifold $Z \subset X$, we identify Z with a submanifold $Z' = w \times Z$ in $\mathbf{S}^1 \times X$, where w is any point in $\mathbf{S}^1$. Z' has even codimension in $\mathbf{S}^1 \times X$, and by a construction explained in the next section, a brane wrapped on Z' determines an element of $\mathrm{K}(\mathbf{S}^1 \times X)$. This element is trivial when restricted to X (that is, to $w' \times X$ for any $w' \in \mathbf{S}^1$). By a more full study of brane-antibrane creation and annihilation, one expects to show that two Type IIA brane configurations on X are equivalent if they determine the same element of $\mathrm{K}(\mathbf{S}^1 \times X)$. The subgroup of $\mathrm{K}(\mathbf{S}^1 \times X)$ consisting of elements that are trivial on X is called $\mathrm{K}^1(X)$.[7] For application to Type IIA, we must consider the subgroup $\widetilde{\mathrm{K}}^1(X) = \widetilde{\mathrm{K}}(\mathbf{S}^1 \times X)$ (since we have no physical interpretation for tenbranes wrapping $\mathbf{S}^1 \times X$!), and we also want a compact support condition that generally makes $\widetilde{\mathrm{K}}^1$ and K^1 equivalent. Generally, then, D-brane charges of Type IIA are classified by $\mathrm{K}^1(X)$, with an appropriate compact support condition.

4. Incorporating p-branes with $p < 9$

In the last section, we saw (with certain assumptions about brane annihilation) that Type IIB configurations of ninebranes, modulo deformation, are classified by $\mathrm{K}(X)$. We also explained the analogs for Type I and Type IIA. In this section, we will show that the charges are still classified in the same way if one relaxes the restriction to ninebranes. The basic idea is to exploit a construction used by Sen [10] to interpret p-branes of $p < 9$ as bound states of brane-antibrane pairs of higher dimension. In the discussion, we assume that all spacetime dimensions are much larger than the string scale. If this discussion is relaxed, one will meet new stringy phenomena. We also assume that the Neveu-Schwarz three-form field H vanishes (at least topologically); it is incorporated in section 5.

4.1 Review of Sen's construction

We first review Sen's basic construction of a p-brane as a bound state of a $p+2$-brane and a coincident $p+2$-antibrane. First we work in $\mathbf{R}^{10}$, without worrying about effects of spacetime topology.

---

homotopy groups $\pi_i(SO(32))$ for $i \leq 9$ and the relations among them. These homotopy groups are in the "stable range", and one can show that $SO(32)$ bundles on X are classified by $\widetilde{\mathrm{KO}}(X)$.

[6] It is tempting to believe that the circle that enters here is related to the circle used in relating Type IIA to M-theory, but I do not know a precise relation.

[7] By Bott periodicity, $\mathrm{K}(X)$ and $\mathrm{K}^1(X)$ are the only complex K-groups of X. So we don't need more Type II string theories that would use more K-groups!

We consider an infinite $p+2$ brane-antibrane pair stretching over an $\mathbf{R}^{p+3} \subset \mathbf{R}^{10}$. On the brane-antibrane pair, there is a $U(1) \times U(1)$ gauge field, with a tachyon field T of charges $(1, -1)$. We consider a "vortex" in which T vanishes on a codimension two subspace $\mathbf{R}^{p+1} \subset \mathbf{R}^{p+3}$, which will be interpreted as the p-brane worldvolume. We suppose that T approaches its vacuum expectation value at infinity, up to gauge transformation. Since T is a complex field, it can have a "winding number" around the codimension two locus on which it vanishes, or equivalently, at infinity. The basic case is the case that the "winding number" is 1. T breaks $U(1) \times U(1)$ to $U(1)$. To keep the energy per unit p-brane volume finite, there must be a unit of magnetic flux in the broken $U(1)$. Because of this magnetic flux, the system has a p-brane charge of 1, as in [25]. Its $(p+2)$-brane charge cancels, of course, between the brane and antibrane. With the tachyon close to its vacuum expectation value except close to the core of the vortex, the system looks like the vacuum everywhere except very near the locus where T vanishes. Since this locus carries unit p-brane charge, it seems that a p-brane has been realized as a configuration of a $(p+2)$-brane-antibrane pair.

How would we generalize this to exhibit a p-brane as a configuration of $p+2k$-branes and antibranes for $k > 1$? One way to do this is to repeat the above construction k times. We first make a p-brane as a bound state of a $(p+2)$-brane and antibrane. Then, we make the $(p+2)$-brane and antibrane each from a $(p+4)$-brane-antibrane pair. So at this stage, the p-brane is built from two $(p+4)$-brane-antibrane pairs. After $k-2$ more such steps, we get a p-brane built from 2^{k-1} pairs of $(p+2k)$-branes and antibranes.

To exhibit the symmetries more fully and for applications below, it helps to make this construction "all at once" and not stepwise. For this, we consider in general a collection of many $(p+2k)$-brane-antibrane pairs, say n such pairs for some sufficiently large n. The branes carry a $U(n) \times U(n)$ gauge symmetry under which the tachyon field T transforms as $(\mathbf{n}, \overline{\mathbf{n}})$. In vacuum, T breaks $U(n) \times U(n)$ down to a diagonal $U(n)$.[8] The gauge orbit of values of T with minimum energy is hence a copy of $U(n)$.

To make a p-brane, we want T to vanish in codimension $2k$ (on an $\mathbf{R}^{p+1} \subset \mathbf{R}^{p+2k+1}$) and to approach its vacuum orbit at infinity, with a non-zero topological "twist" around the locus on which T vanishes. Such configurations are classified topologically by $\pi_{2k-1}(U(n))$. According to Bott periodicity, this group equals $\mathbf{Z}$ for all sufficiently large n. This copy of $\mathbf{Z}$ will label the possible values of p-brane charge. The value $n = 2^{k-1}$ is suggested by the above stepwise construction, and indeed for this value one can give a particular simple and — as we will see — useful description of the generator of $\pi_{2k-1}(U(n))$. Let S_+ and S_- be the positive and negative chirality

[8] The possibility of separating the brane-antibrane pairs indicates that the eigenvalues of T are all equal in vacuum, so that in vacuum T breaks $U(n) \times U(n)$ to $U(n)$, rather than a subgroup. As noted in [12] and above, there is a puzzle here, namely how to think about the fate of the diagonal $U(n)$ that is not broken by T.

spinor representations of $SO(2k)$; they are of dimension 2^{k-1}. Let $\vec{\Gamma} = (\Gamma_1, \ldots, \Gamma_{2k})$ be the usual Gamma matrices, regarded as maps from S_- to S_+. If $\vec{x} = (x_1, \ldots, x_{2k})$ is an element of $\mathbf{S}^{2k-1}$ (that is, a $2k$-vector with $\vec{x}^2 = 1$), then we define the tachyon field by

$$T(\vec{x}) = \vec{\Gamma} \cdot \vec{x}\,. \tag{4.1}$$

It has winding number 1 and (according to sec. 2.13 of [26]) generates $\pi_{2k-1}(U(2^{k-1}))$.

That the p-brane charge of this configuration is 1 and all higher (and lower) charges vanish can be verified by using the formulas for brane charges induced by gauge fields, or in a more elementary way by verifying that the "all at once" construction (4.1) is equivalent to the stepwise construction that we described first.

Since this configuration has p-brane charge 1 and looks like the vacuum except near $\vec{x} = 0$, we assume, in the spirit of Sen's constructions, that this configuration describes a p-brane.

We now wish to place this construction in a global context. The goal is to show that, globally, brane charge in a spacetime X can always be described by a configuration of 9-branes and $\overline{9}$-branes and so is classified as in section 3. The technical arguments that follow can be found in [26]; physics will intrude only when we discuss the anomaly.

4.2 Global version

We first consider Sen's original construction (codimension two) in a global context. We let Z be a closed submanifold of spacetime, of dimension $q = p + 1$, and we suppose that Z is contained in Y, a submanifold of spacetime of dimension $q + 2$. We assume here and in section 4.3 that Z and Y are orientable, since Type II branes can only wrap on orientable manifolds. Then one can define a complex line bundle $\mathcal{L}$ over Y, and a section s of $\mathcal{L}$ that vanishes precisely along Z, with a simple zero. Moreover, one can put a metric on $\mathcal{L}$ such that, except in a small neighborhood of Z, s has fixed length.

Now, consider a system consisting of a $(p+2)$-brane-antibrane pair, wrapped on Y. We place on the brane a $U(1)$ gauge field that is a connection on $\mathcal{L}$; its p-brane charge is that of a p-brane wrapped on Z. We place on the antibrane a trivial $U(1)$ gauge field, with vanishing p-brane charge. The brane-antibrane system thus has vanishing $(p+2)$-brane charge and p-brane charge the same as that of a p-brane on Z. This suggests that the system could be deformed to a system consisting just of a p-brane wrapped on Z.

As evidence for this, we note that the tachyon field of the brane-antibrane pair, because it has charges $(1, -1)$ under the $U(1) \times U(1)$ that live on the brane and antibrane, should be a section of $\mathcal{L}$. Hence we can take

$$T = c \cdot s\,, \tag{4.2}$$

with c a constant chosen so that far from Z, $|T|$ is equal to its vacuum expectation value. In future, we will generally omit constants analogous to c, to avoid cluttering the formulas. The basic assumptions about brane-antibrane annihilation then suggest that with this choice of T, the system is in a vacuum state except near Z and can be described by a p-brane wrapped on Z.

Incorporation of lower charges. A fuller description actually requires the following generalization. Note that a p-brane wrapped on Z has in general in addition to its p-brane charge also r-brane charges with $r = p-2, p-4, \ldots$. Moreover, these depend on the choice of a line bundle $\mathcal{M}$ on Z. Thus, to fully describe all states with a p-brane wrapped on Z in terms of states of a brane-antibrane pair wrapped on Y, we need a way to incorporate $\mathcal{M}$ in the discussion.

If $\mathcal{M}$ extends over Y, we incorporate it in the above discussion just by placing the line bundle $\mathcal{L}\otimes\mathcal{M}$ on the $p+2$-brane and the line bundle $\mathcal{M}$ on the $\overline{p+2}$-brane. The tachyon field T, given its charges $(1,-1)$, is a section of $(\mathcal{L}\otimes\mathcal{M})\otimes\mathcal{M}^{-1} = \mathcal{L}$, so we can take $T = s$ and flow (presumably) to a configuration containing only a p-brane wrapped on Z. The r-brane charges with $r < p$ now depend on $\mathcal{M}$ in a way that has a simple interpretation: on the p-brane worldvolume there is a $U(1)$ gauge field with line bundle $\mathcal{M}$.

More generally, however, $\mathcal{M}$ may not extend over Y. To deal with this case, we need to use another of the basic constructions of K-theory. First we describe it in mathematical terms. Let Z be a submanifold of a manifold Y, and Z' a tubular neighborhood of Z in Y (this means that we pick a suitable metric on Y, and let Z' consist of points of distance $< \epsilon$ from Z, for some small ϵ). Let $\overline{Z}$ be the closure of Z' (the points of distance $\leq \epsilon$ from Z) and Z^* its boundary (the points of distance precisely ϵ). Suppose that E and F are two bundles over Z of the same rank (in our example so far, they are line bundles, $E = \mathcal{L}\otimes\mathcal{M}$ and $F = \mathcal{M}$), so that the pair (E,F) defines an element of $\mathrm{K}(Z)$. Pull E and F back to $\overline{Z}$, so that (E,F) defines an element of $\mathrm{K}(\overline{Z})$. The tachyon field T, which is a section of $E\otimes F^*$, can be regarded as a bundle map

$$T : F \to E\,. \tag{4.3}$$

Suppose that T is a tachyon field on $\overline{Z}$ which (when viewed in this way as a bundle map) is an isomorphism (an invertible map) if restricted to Z^*. Then from this data, one can construct an element of $\mathrm{K}(Y)$.[9] The construction is made as follows. Let $Y' = Y - Z'$; thus Y' consists of Z^* and its "exterior" in Y. If we could extend the bundle F from Z^* over all of Y' then F would be defined over all of Y (since it is defined already on $\overline{Z}$). Since E is isomorphic to F on Z^* (via T), we

[9]Moreover, the construction gives a natural map from elements of $\mathrm{K}(\overline{Z})$ trivialized on Z^* to $\mathrm{K}(Y)$, in the sense that the image of (E,F) with bundle map T is, for any H, the same as the image of $(E\oplus H, F\oplus H)$, with bundle map $T\oplus 1$.

JHEP12(1998)019

could extend it over Y' by declaring that it is isomorphic to F over Y'. The pair (E, F) of bundles on Y then give the desired element of $\mathrm{K}(Y)$.

If F does not extend over Y', one proceeds as follows. By a standard lemma in K-theory (Corollary 1.4.14 in [15]), there is a bundle H over Z such that $F \oplus H$ is trivial over Z, and hence is trivial when pulled back to $\overline{Z}$. Replacing E, F, and T by $E \oplus H$, $F \oplus H$, and $T \oplus 1$, we can extend $F \oplus H$ over Y (as a trivial bundle), and extend $E \oplus H$ by setting it equal to $F \oplus H$ over Y'. The pair $(E \oplus H, F \oplus H)$ then give the desired element of $\mathrm{K}(X)$. Note that $E \oplus H$ and $F \oplus H$ are isomorphic over Y' but not over Y.

This construction is precisely what we need to express in terms of $(p+2)$-branes on Y a p-brane on Z that supports a line bundle $\mathcal{M}$. We find a bundle H over Z such that $\mathcal{M} \oplus H$ is trivial (and so extends over Y). $\mathcal{L} \otimes \mathcal{M} \oplus H$ is extended over Y using the fact that (via $T \oplus 1$) it is isomorphic to $\mathcal{M} \oplus H$ away from Z. Then we consider a collection of $(p+2)$-branes on Y with gauge bundle $\mathcal{L} \otimes \mathcal{M} \oplus H$, and $(\overline{p+2})$-branes on Y with gauge bundle $\mathcal{M} \oplus H$. The number of branes of each kind is 1 plus the rank of H. The tachyon field is $T \oplus 1$ near Z, and is in the gauge orbit of the vacuum outside of Z'. The system thus describes, under the usual assumptions, a p-brane on Z with gauge bundle $\mathcal{M}$.

In a similar fashion, we could have started with any collection of n p-branes wrapped on Z, with $U(n)$ gauge bundle $\mathcal{W}$, and expressed it in terms of a collection of $(p+2)$-branes and antibranes on Y. One pulls back $\mathcal{W}$ to $\overline{Z}$, uses the tachyon field $\widetilde{T} = T \otimes 1$ to identity $\mathcal{W}$ with $\mathcal{L} \otimes \mathcal{W}$ on the boundary of $\overline{Z}$, and then uses the pair of bundles $(\mathcal{L} \otimes \mathcal{W}, \mathcal{W})$ to determine a class in $\mathrm{K}(Y)$. Such a class is, finally, interpreted in terms of a collection of branes and antibranes wrapped on Y. The p-brane charge is n; the r-brane charges for $r < p$ depend on $\mathcal{W}$.

4.3 Spinors and the anomaly

Specializing to the case that Y coincides with the spacetime manifold X and Z is of codimension two in X, this construction shows that whatever can be done with sevenbranes can be done with ninebranes. We now wish to show that brane wrapping on a submanifold Z of codimension greater than two can likewise be expressed globally in terms of ninebranes.

Under favorable conditions, there might be a chain of embeddings $Z \subset Z' \subset \ldots \subset X$, with codimension two at each stage. Then we could inductively use the construction already explained. In general, however, such a chain of embeddings will not exist globally. Instead, we will use spinors, building on facts explained at the end of section 4.1.

Let N be the normal bundle to Z in X. If Z has codimension $2k$, then the structure group of N is $SO(2k)$. We suppose first that N is a spin bundle, which means that $w_2(N) = 0$ and that there are bundles $\mathcal{S}_+$, $\mathcal{S}_-$ associated to N by using the positive and negative chirality spin representations of $SO(2k)$.[10] We consider a system of 2^{k-1} 9-branes, and the same number of $\overline{9}$ branes, with the gauge bundles on them, near Z, being $\mathcal{S}_+$ and $\mathcal{S}_-$.[11] This system has all r-brane charges zero for $r > p$ (as they cancel between the branes and antibranes), while its p-brane charge is that of a single brane wrapped on Z.

The tachyon field T from the 9-$\overline{9}$ sector should be a map from $\mathcal{S}_-$ to $\mathcal{S}_+$. The basic such maps are the Dirac Gamma matrices Γ. We identify a tubular neighborhood Z' of Z in X with the vectors in N of length < 1, and for $x \in Z'$, we write

$$T = \vec{\Gamma} \cdot \vec{x}\,. \tag{4.4}$$

T gives a unitary isomorphism between $\mathcal{S}_-$ and $\mathcal{S}_+$ on the boundary of Z' (since $\vec{\Gamma}\cdot\vec{x}$ is unitary if $\vec{x}$ is a unit vector), so (after scaling by a constant c, which we suppress) T on the boundary of Z' is in the gauge orbit of the vacuum. If, therefore, we can extend this configuration over X, keeping T equal to its vacuum expectation value, then we will get a system of 9-branes and $\overline{9}$-branes that represents a single p-brane wrapped on Z. As in the discussion at the end of section 4.2, the configuration we have described can be extended over X if the bundle $\mathcal{S}_-$ so extends. Otherwise, we pick a suitable H such that $\mathcal{S}_-\oplus H$ extends, and replace $(\mathcal{S}_+,\mathcal{S}_-)$ by $(\mathcal{S}_+\oplus H, \mathcal{S}_-\oplus H)$ and T by $T\oplus 1$.

More generally, to describe a p-brane on Z with a line bundle $\mathcal{M}$, we use the 9-brane configuration

$$(\mathcal{M}\otimes\mathcal{S}_+, \mathcal{M}\otimes\mathcal{S}_-)\,, \tag{4.5}$$

or still more generally $(\mathcal{M}\otimes\mathcal{S}_+\oplus H, \mathcal{M}\otimes\mathcal{S}_-\oplus H)$, with H chosen so that these bundles extend over X. The tachyon field is still $T = \vec{\Gamma}\cdot\vec{x}$ (or $\vec{\Gamma}\cdot\vec{x}\oplus 1$) in a neighborhood of Z, and lies in its vacuum orbit in the complement of this neighborhood.

The $Spin_c$ case. So far we have assumed that the normal bundle N to Z in spacetime is spin, $w_2(N) = 0$. What if it is not?

If instead of being spin, N admits a Spin_c structure, then we can proceed much as before. N not being spin means the following. If we cover X with open sets W_i, then the would-be transition functions w_{ij} of $\mathcal{S}_+$ (or similarly $\mathcal{S}_-$) on $W_i\cap W_j$ obey

$$w_{ij}w_{jk}w_{ki} = \phi_{ijk}\,, \tag{4.6}$$

where $\phi_{ijk} = \pm 1$. The ϕ_{ijk} are a two-cocycle (with values in $\{\pm 1\}$) defining $w_2(N) \in$

[10] N may have different spin structures. In fact, since to do Type IIB theory on X, X is endowed with a spin structure, a choice of spin structure on N is equivalent to a choice of spin structure on Z. The K-theory class determined by Z may in general depend on its spin structure — or more generally, on its Spin_c structure, as described below.

[11] We tacitly assume for the moment that $\mathcal{S}_\pm$ extend over X and postpone the technicalities that arise when this is not so.

$H^2(Z, \mathbf{Z}_2)$.[12] N being Spin_c means that there is a line bundle $\mathcal{L}$ over Z, with the following property. Let f_{ij} be transition functions on $W_i \cap W_j$ defining $\mathcal{L}$. The Spin_c property arises if a square root $\mathcal{L}^{1/2}$ of $\mathcal{L}$ does not exist as a line bundle, but is obstructed by the same cocycle that obstructs existence of $\mathcal{S}_\pm$. This happens if putative transition functions $g_{ij} = \pm\sqrt{f_{ij}}$ of $\mathcal{L}^{1/2}$ (with suitable choices of the signs) obey $g_{ij}g_{jk}g_{ki} = \phi_{ijk}$. In this case, the cocycle cancels out in the transition functions $g_{ij}w_{ij}$ of the vector bundles $\mathcal{L}^{1/2} \otimes \mathcal{S}_\pm$, and these objects (which are sometimes called Spin_c bundles) exist as honest vector bundles, even though the factors $\mathcal{L}^{1/2}$ and $\mathcal{S}_\pm$ do not separately have that status.

Notice that such an $\mathcal{L}$, if it exists, will generally be far from unique. We could pick any line bundle $\mathcal{M}$ over Z and replace $\mathcal{L}$ by $\mathcal{M}^2 \otimes \mathcal{L}$; this maps the Spin_c bundles $\mathcal{L}^{1/2} \otimes \mathcal{S}_\pm$ to $\mathcal{M} \otimes \mathcal{L}^{1/2} \otimes \mathcal{S}_\pm$, which certainly exist if and only if $\mathcal{L}^{1/2} \otimes \mathcal{S}_\pm$ do. One way to characterize the allowed $\mathcal{L}$'s is as follows. Define $x \in H^2(Z, \mathbf{Z})$ by $x = c_1(\mathcal{L})$. Then modulo 2, x is invariant under $\mathcal{L} \to \mathcal{M}^2 \otimes \mathcal{L}$, and it can be shown that existence, in the above sense, of $\mathcal{L}^{1/2} \otimes \mathcal{S}_\pm$ is equivalent to

$$x \cong w_2(N) \bmod 2\,. \tag{4.7}$$

The criterion that a Spin_c structure exists can be stated as follows. Consider the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}_2 \to 0\,, \tag{4.8}$$

where the first map is multiplication by 2 and the second is reduction modulo 2. The associated long exact sequence of cohomology groups reads in part

$$\cdots \to H^2(Z, \mathbf{Z}) \to H^2(Z, \mathbf{Z}_2) \xrightarrow{\beta} H^3(Z, \mathbf{Z}) \to \cdots \tag{4.9}$$

The image of $w_2(N) \in H^2(Z, \mathbf{Z}_2)$ under the map that has been called β in (4.9) is an element of $H^3(Z, \mathbf{Z})$ called $W_3(N)$. (β is called the Bockstein homomorphism.) Exactness of the sequence (4.9) implies that $w_2(N)$ can be lifted to $x \in H^2(Z, \mathbf{Z})$ — and hence N is Spin_c — if and only if $W_3(N) = 0$.

Returning now to our overall problem of interpreting a brane wrapped on Z in terms of an element of $\mathrm{K}(X)$, if the bundle N is Spin_c we can proceed precisely as we did in the spin case. The bundles $(\mathcal{L}^{1/2} \otimes \mathcal{S}_+, \mathcal{L}^{1/2} \otimes \mathcal{S}_-)$, with the tachyon field T still defined as in (4.4), determine the desired element of $\mathrm{K}(X)$ that represents a brane wrapped on Z. The possibility of tensoring $\mathcal{L}^{1/2}$ with an arbitrary line bundle $\mathcal{M}$ just corresponds to the fact that the brane wrapped on Z could support an arbitrary line bundle.

[12]The cocycle property can be proved from (4.6) as follows. First rewrite this formula as $w_{ij}w_{jk} = \phi_{ijk}w_{ik}$. Now consider the product $w_{ij}w_{jk}w_{kl}$. This product can be evaluated by associativity as $(w_{ij}w_{jk})w_{kl} = \phi_{ijk}\phi_{ikl}w_{il}$, or $w_{ij}(w_{jk}w_{kl}) = \phi_{ijl}\phi_{jkl}w_{il}$. Comparing these gives the cocycle relation $\phi_{ijk}\phi_{ikl} = \phi_{ijl}\phi_{jkl}$.

The topological obstruction. Now we come to a key point. What if the normal bundle N is *not* Spin_c?

There seems to be a puzzle. If N is not Spin_c, a brane wrapped on Z does not determine a K-theory class. This appears to contradict the relation between branes and K-theory.

The answer, surprisingly, is that if N is not Spin_c, a brane cannot be wrapped on Z. This follows from a topological obstruction to brane wrapping that was observed in a particular situation in [13] and has been extracted from world-sheet global anomalies [14]. The obstruction in question appears in eqn. (3.13) of [13]. (The $[H]$ term in that equation can be dropped, since we are assuming at the present that the cohomology class of the Neveu-Schwarz three-form is zero.) Let w_i, $i = 1, 2, \dots$ denote Stieffel-Whitney classes. In particular, let $w_i(Z)$ be the Stieffel-Whitney classes of the tangent bundle of Z. Let $W_3(Z) = \beta(w_2(Z))$ (with β being the Bockstein). Equation (3.13) of [13] says that a brane can wrap on Z if and only if $W_3(Z) = 0$.

Using the fact that X is spin, $w_1(X) = w_2(X) = 0$, and that Z is orientable, $w_1(Z) = 0$, a standard argument[13] gives $w_2(N) = w_2(Z)$, and hence $W_3(N) = W_3(Z)$. This is compatible with the idea that the charges carried by branes are measured by K-theory classes. If $W_3(N) \neq 0$, then the construction of a K-theory class that would be carried by a brane wrapped on Z fails, as we have seen above, but this presents no problem since such a wrapped brane does not exist.

Even when $W_3(N)$ is nonzero, it is possible to have a configuration consisting of *several* branes wrapped on Z, supporting suitable gauge fields. The gauge bundle $\mathcal{W}$ on Z must not be a true vector bundle; its transition functions must close up to $\pm$ signs in just such a way that $\mathcal{W} \otimes \mathcal{S}_\pm$ exists as a vector bundle. The most obvious choice is $\mathcal{W} = \mathcal{S}_+$ (or $\mathcal{S}_-$); the bundles $\mathcal{S}_+ \otimes \mathcal{S}_\pm$ exist as they can be expressed in terms of differential forms. If Z is of codimension $2k$ in X, then the rank of $\mathcal{S}_+$ is 2^{k-1}. The K-theory class associated with $(\mathcal{S}_+ \otimes \mathcal{S}_+, \mathcal{S}_+ \otimes \mathcal{S}_-)$ and the usual tachyon field (4.4) describes a configuration of 2^{k-1} branes wrapped on Z, supporting a "gauge bundle" $\mathcal{S}_+$. In some cases with $W_3(N) \neq 0$, it is possible to find a more economical solution with a smaller (but even) number of branes wrapped on Z.

I will not in this paper discuss the analogous issues for Type IIA, except to note that having come to this point, the reader may now find the comments in the last paragraph of section 3 to be clearer. We move on next to discuss some analogous questions for Type I superstrings.

4.4 Spinors and Type I branes

In studying Type I superstrings, we begin with the D-string. We wish to exhibit it as a bound state of 9-branes and $\overline{9}$-branes.

[13] The multiplicativity of the total Stieffel-Whitney class in direct sums gives $(1+w_1(X)+w_2(X)+\dots) = (1 + w_1(Z) + w_2(Z) + \dots)(1 + w_1(N) + w_2(N) + \dots)$. With $0 = w_1(X) = w_2(X) = w_1(Z)$, we get $0 = w_1(N)$ and $w_2(N) = w_2(Z)$.

We want to describe a D-string located at $x^1 = \ldots = x^8 = 0$ in $\mathbf{R}^{10}$; its world-volume Z is parametrized by x^0 and x^9.

The group of rotations keeping Z fixed is $K = SO(8)$. K rotates the eight-vector $\vec{x} = (x^1, \ldots, x^8)$. The two spinor representations of K are both eight-dimensional; we call them S_+ and S_-. We regard the Γ-matrices Γ^i as maps from S_- to S_+.

We consider a configuration of eight 9-branes and eight $\overline{9}$-branes. We take the gauge bundles on these branes to be trivial, but we take the rotation group K to act on the Chan-Paton labels, with the rank eight bundle of the 9-branes transforming as S_+, and the $\overline{9}$-brane bundle transforming as S_-.

For the tachyon field, we take

$$T(\vec{x}) = f(|\vec{x}|)\vec{\Gamma} \cdot \vec{x}\,, \tag{4.10}$$

with f a function that is 1 near $|\vec{x}| = 0$, and $c/|\vec{x}|$ for $|\vec{x}| \to \infty$, with a suitable constant c. T/c is an orthogonal matrix for $|\vec{x}| \to \infty$; c is chosen so that T lies in the gauge orbit of the vacuum. In the spirit of Sen's constructions, we expect that this configuration is equivalent to the vacuum except near $\vec{x} = 0$. Thus it describes a string localized near $\vec{x} = 0$. In future we will generally omit the $f(|\vec{x}|)$ factor to avoid clutter.

Note that with the chosen action of $SO(8)$ on the gauge bundles, the tachyon field is $SO(8)$-invariant, so the construction is manifestly $SO(8)$-invariant. Moreover, if we restore the extra 32 9-branes that are needed for Type I tadpole cancellation, then $SO(32)$ gauge symmetry of these 9-branes is just a spectator in this construction; the construction is manifestly $SO(32)$-invariant.

In this construction, the gauge field on the branes must be chosen so that T is covariantly constant near infinity. Since $\vec{x} \to \vec{\Gamma} \cdot \vec{x}/|x|$ is the generator of $\pi_7(SO(8)) = \mathbf{Z}$, the configuration that we have built is a "gauge string", in the sense of section 2. However, now we have used extra brane-antibrane pairs to enlarge the gauge group, and have made the construction in a manifestly $SO(32)$-invariant way.

Now, as in section 4.3, we would like to make this construction globally. This involves no essential novelty compared to Type IIB, except perhaps for the fact that there is no topological anomaly to worry about. We consider a two-surface Z in a Type I spacetime X. In Type I superstring theory, X is spin, so $w_1(X) = w_2(X) = 0$. To wrap a D-string on Z, Z must be orientable, so $w_1(Z) = 0$, and hence (as orientable two-manifolds are spin) $w_2(Z) = 0$. We pick a spin structure on X (since Type I requires a spin structure) and on Z (D-string wrapping on a two-cycle is expected to depend on a choice of spin structure on the two-cycle). Let N be the normal bundle to Z in X. Since $(1 + w_1(X) + w_2(X) + \ldots) = (1 + w_1(Z) + w_2(Z) + \ldots)(1 + w_1(N) + w_2(N) + \ldots)$, one has $w_1(N) = w_2(N) = 0$, so spin bundles $\mathcal{S}_-$ and $\mathcal{S}_+$ (derived from N using the spin representations of $SO(8)$) exist. More specifically, the chosen spin structures for X and Z determine in a natural way a spin structure for N.

Taking eight 9-branes whose gauge bundle near Z is identified with $\mathcal{S}_+$, and eight $\overline{9}$-branes with gauge bundle near Z identified with $\mathcal{S}_-$, and a tachyon field that looks like $T = \vec{\Gamma} \cdot \vec{x}$ near Z, we express the D-string wrapped on Z in terms of eight 9-$\overline{9}$ pairs. An interesting point is that as Z is two-dimensional and $w_1(\mathcal{S}_-) = w_2(\mathcal{S}_-) = 0$, $\mathcal{S}_-$ is actually trivial along Z, and hence can be extended over X as a trivial bundle. Hence, in contrast to Type IIB, there is no need to "stabilize" by adding extra 9-$\overline{9}$ pairs; eight of them is always enough.

There was no need in this construction to assume that Z is connected. So any collection of D-strings, at least if they are disjoint, can be represented by a configuration of eight 9-$\overline{9}$ pairs. There is no need to introduce eight more pairs for every D-string! Since D-strings are equivalent to perturbative heterotic strings, this is close to saying that the second quantized Fock space of perturbative heterotic strings can be described by configurations of eight 9-$\overline{9}$ pairs.

Fivebranes. We will briefly discuss fivebranes in a similar spirit.

The basic local fact making it possible to interpret fivebranes as ninebrane configurations is that a Type I fivebrane is equivalent to an instanton on the ninebranes that fill the vacuum [27]. If one nucleates extra 9-$\overline{9}$ pairs, there is more flexibility. Consider a system of four 9-$\overline{9}$ pairs, so that the 9-brane Chan-Paton group is $SO(4) = SU(2) \times SU(2)$. Place an instanton of instanton number 1 in one of the $SU(2)$'s. This makes a configuration whose fivebrane number is equal to 1. (We picked four 9-$\overline{9}$ pairs as it is the smallest number for which the fivebrane number can be 1.) With a suitable tachyon field (very similar in fact to what is discussed in [10] in showing that Type I D-strings can be made from fivebranes), this should be equivalent to a fivebrane.

I leave it to the reader to analyze this construction globally, and show that there is no obstruction to similarly making a fivebrane that is wrapped on any spin six-cycle out of 9-$\overline{9}$ pairs.

The $SO(4) \times SO(4)$ gauge symmetry of four 9-$\overline{9}$ pairs is broken to a diagonal $SO(4)$ by the tachyon field and to $SU(2) = \mathrm{Sp}(1)$ by the instanton. This is the usual $\mathrm{Sp}(1)$ gauge symmetry of a Type I fivebrane. The $\mathrm{Sp}(1)$ gauge symmetry is associated with the following mathematical fact. Type I fivebrane charge takes values in $\mathrm{KO}(\mathbf{S}^4)$ (or equivalently $\mathrm{KO}(\mathbf{R}^4)$ with compact support, the $\mathbf{R}^4$ parametrizing here the directions normal to the fivebrane). Type IIB fivebrane charge likewise takes values in $\mathrm{K}(\mathbf{S}^4)$. The groups $\mathrm{KO}(\mathbf{S}^4)$ and $\mathrm{K}(\mathbf{S}^4)$ are both isomorphic to $\mathbf{Z}$. But the natural map from $\mathrm{KO}(\mathbf{S}^4)$ to $\mathrm{K}(\mathbf{S}^4)$ (defined by forgetting that the bundles are real), is multiplication by 2. This is because the generator of $\mathrm{K}(\mathbf{S}^4)$ is an $SU(2)$ instanton field, which is a pseudoreal bundle and must be embedded in $SO(4)$ (as we did two paragraphs ago) if one wants to make it real. The embedding in $SO(4)$ doubles the charge, so the fivebrane charge of a Type I fivebrane is twice that of a Type IIB fivebrane.

This completes the demonstration that the Type I configurations built from the usual supersymmetric branes (p-branes for $p = 1, 5, 9$) represent classes in $\mathrm{KO}(X)$. However, there is more to say. The relation to $\mathrm{KO}(X)$ suggests that Type I should also have, for example, zerobranes — associated with $\mathrm{KO}(\mathbf{S}^9) = \mathbf{Z}_2$ — and -1-branes — associated with $\mathrm{KO}(\mathbf{S}^{10}) = \mathbf{Z}_2$. Concretely, the assertions that $\mathrm{KO}(\mathbf{S}^9) = \mathbf{Z}_2$ and that $\mathrm{KO}(\mathbf{S}^{10}) = \mathbf{Z}_2$ are equivalent to the assertions that $\pi_8(SO(k)) = \pi_9(SO(k)) = \mathbf{Z}_2$ for sufficiently large k. We examined topological defects associated with these homotopy groups in section 2; we will now reexamine them in light of our experience with K-theory.

4.5 The Type I zerobrane and -1-brane

For $n = 9$ or 10, we will think of $\mathrm{KO}(\mathbf{S}^n)$ as $\mathrm{KO}(\mathbf{R}^n)$ with compact support. An element of $\mathrm{KO}(\mathbf{R}^n)$ with compact support is described by giving two $SO(k)$ bundles E and F over $\mathbf{R}^n$ (for some k), with a bundle map $T : F \to E$ that is an isomorphism near infinity. The physical interpretation, as we have seen, is that E is the Chan-Paton bundle of k 9-branes, F the Chan-Paton bundle of k $\overline{9}$-branes, and T the 9-$\overline{9}$ tachyon field. Since $\mathbf{R}^n$ is contractible, the bundles E and F are trivial; the topology is all in the "winding" of T near infinity.

The standard mathematical descriptions (see [26]) of generators of $\mathrm{KO}(\mathbf{R}^9)$ and $\mathrm{KO}(\mathbf{R}^{10})$ with compact support are similar to what we have seen already in describing the more familiar supersymmetric branes of Type II and Type I superstring theory. In each case, we take E and F to be trivial bundles on which the rotation group of $\mathbf{R}^9$ or $\mathbf{R}^{10}$ acts in the spin representation. Since we want KO theory, we must use real spin representations.

In the case of $\mathbf{R}^9$, we need the spinor representation of $SO(9)$. There is only one such irreducible representation S. It is real and of dimension 16, so we consider the case that E and F are 16-dimensional and transform under rotations like S. The tachyon field is given by the familiar formula:

$$T(x) = \sum_{\mu=1}^{9} \Gamma_\mu x^\mu \,, \tag{4.11}$$

with x^μ, $\mu = 1, \ldots, 9$ the coordinates of $\mathbf{R}^9$, and Γ_μ the Γ matrices.[14] This configuration is manifestly $SO(9)$-invariant, with the indicated action of $SO(9)$ on the Chan-Paton bundles E and F.

Now to compare this to the description of the Type I zerobrane given in [10], we want to make an $8+1$-dimensional split of the coordinates and Gamma matrices. We pick an $SO(8)$ subgroup of $SO(9)$, under which the x^μ break up as $\vec{x}, x^9$, with $\vec{x} = (x^1, \ldots, x^8)$ and x^9 the last coordinate. The representation S of $SO(9)$ breaks

[14] As earlier, the formula for $T(x)$ should more properly be $T = f(|x|)\Gamma_\mu x^\mu$, with f constant for small $|x|$ and $f \sim 1/|x|$ for large $|x|$. To keep the formulas simple, we omit this factor.

up under $SO(8)$ as $S_+ \oplus S_-$, with S_+ and S_- the positive and negative chirality spinor representations of $SO(8)$, which are of course both real. We write the $SO(8)$ Gamma matrices as $\vec{\Gamma} : S_- \to S_+$, $i = 1, \ldots, 8$, and their transposes $\vec{\Gamma}^T : S_+ \to S_-$. In a basis in which we write the $SO(9)$ spinors as

$$S = \begin{pmatrix} S_+ \\ S_- \end{pmatrix}, \tag{4.12}$$

the tachyon field of equation (4.11) is then

$$T = \begin{pmatrix} x^9 & \vec{\Gamma} \cdot \vec{x} \\ \vec{\Gamma}^T \cdot \vec{x} & -x^9 \end{pmatrix}. \tag{4.13}$$

If we make a change of basis on the $\overline{9}$-branes by the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.14}$$

then the tachyon field is transformed to

$$T = \begin{pmatrix} \vec{\Gamma} \cdot \vec{x} & x^9 \\ -x^9 & \vec{\Gamma}^T \cdot \vec{x} \end{pmatrix}. \tag{4.15}$$

This formula has a nice intuitive interpretation. Suppose first that we neglect the off-diagonal blocks in (4.15). Then the system splits up into two decoupled systems each containing eight 9-$\overline{9}$ pairs. The first set of eight pairs has tachyon field

$$T_1 = \vec{\Gamma} \cdot \vec{x} \tag{4.16}$$

and the second has tachyon field

$$T_2 = \vec{\Gamma}^T \cdot \vec{x}\,. \tag{4.17}$$

As we have discussed at the beginning of the present section, T_1 describes a D-string located at $x^1 = \ldots = x^8 = 0$. Since T_2 is made from T_1 by exchanging the 9-branes with $\overline{9}$-branes, T_2 describes an anti D-string located at $x^1 = \ldots = x^8 = 0$. This is precisely the configuration of a coincident D-string and anti D-string coinsidered by Sen [10]. Moreover, the off-diagonal blocks in (4.15) can be understood as a tachyon field which connects the D-string and anti D-string and is odd under $x^9 \to -x^9$. This is precisely the solitonic configuration of the 1-$\overline{1}$ tachyon field that is described in [10]. So we have made contact with this form of Sen's construction.

In section 6, we will attempt to give a slightly simplified version of the worldsheet description in [7]. As preparation for that, let us notice the following suggestive fact. To describe Type IIB branes of codimension $2k$, we used spinors of $SO(2k)$ of definite chirality. The dimension of the chiral spinors of $SO(2k)$ is 2^{k-1}, and this was the number of 9-$\overline{9}$ pairs used to describe a brane of codimension $2k$.

For the Type I zerobrane, the codimension is 9. To use the same formula as in the other cases, we would set $2k = 9$ and expect the number of 9-$\overline{9}$ pairs to be $2^{k-1} = 8\sqrt{2}$. This does not make sense as it is not an integer. The actual number in the above construction is 16, larger than $8\sqrt{2}$ by a factor of $\sqrt{2}$. We will seek to interpret the extra factor of $\sqrt{2}$ in section 6.

The -1-brane. Now we consider the -1-brane. For this, we must take E and F to be spinor representations of $SO(10)$. Moreover, we must use *real* spinor representations, as we are doing Type I superstrings and KO theory. The group $SO(32)$ has a unique irreducible real spinor representation S; it is 32-dimensional. The -1-brane is described with 32 9-$\overline{9}$ pairs and a tachyon field given by the usual formula $T = \vec{\Gamma} \cdot \vec{x}$.

As preparation for a worldsheet construction discussed in section 6, we note the following. Although the representation S is irreducible over the real numbers, if complexified it decomposes as $S = S_+ \oplus S_-$ where S_+ and S_- are the 16-dimensional complex spinor representations of $SO(10)$ of positive and negative chirality. The tachyon field $T = \vec{\Gamma} \cdot \vec{x}$ of course reverses the chirality. If therefore we were working in Type IIB and all matrices were complex, we would decompose this system as a sum of two subsystems, one with 9-brane and $\overline{9}$-brane representations $(E_1, F_1) = (S_+, S_-)$, and the second with representations $(E_2, F_2) = (S_-, S_+)$. (The tachyon field is $T = \vec{\Gamma} \cdot \vec{x}$, mapping F_1 to E_1 and F_2 to E_2.) The (S_+, S_-) system is the by now familiar description of a Type IIB -1-brane, and the (S_-, S_+) system, which has the chiralities or equivalently the 9-branes and $\overline{9}$-branes reversed, describes similarly a Type IIB anti -1-brane. The orientation projection that reduces Type IIB to Type I acts by complex conjugation, so it exchanges S_+ and S_- and hence exchanges the -1-brane with the anti -1-brane. This information will enable us in section 6 to give a worldsheet description of the Type I -1-brane.

One could also make a $9 + 1$-dimensional or $8 + 2$-dimensional split of the 10-dimensional Gamma matrices, and describe the Type I -1-brane in terms of configurations of zerobranes or onebranes and coincident antibranes, with suitable tachyon fields. We will omit this.

The sevenbrane and eightbrane. Both in section 2 and here, we have focussed our discussion of Type I on the topological objects associated with π_7, π_8, and π_9. What other nonzero homotopy groups are there in a range that is relevant in ten dimensions? π_3 is nonvanishing and is associated with the familiar Yang-Mills instantons, or alternatively with the Type I fivebrane, which we have discussed in section 4.4. The other candidates are

$$\pi_0(O(32)) = \pi_1(O(32)) = \mathbf{Z}_2\,, \tag{4.18}$$

where here we recall that the perturbative gauge group is more nearly $O(32)$ than $SO(32)$.[15] This suggests that one could make in Type I a sevenbrane and an eightbrane, related to $\mathrm{KO}(\mathbf{S}^2)$ and $\mathrm{KO}(\mathbf{S}^1)$ respectively.

As usual we identify $\mathrm{KO}(\mathbf{S}^n)$ with $\mathrm{KO}(\mathbf{R}^n)$ with compact support; and we describe an element of $\mathrm{KO}(\mathbf{R}^n)$ with compact support by giving trivial bundles E, F on $\mathbf{R}^n$ and a tachyon map between them that is invertible at infinity. The formula for the tachyon map is always $T = \vec{\Gamma} \cdot \vec{x}$.

For $n = 1$, there is only one Gamma matrix. We can take it to be the 1×1 unit matrix. The tachyon field is thus

$$T = x^9 \tag{4.19}$$

(times a convergence factor such as $1/\sqrt{1 + (x^9)^2}$), where for convenience we have labeled as x^9 the coordinate on $\mathbf{R}^1$. Thus, the eightbrane is a "domain wall", located at $x^9 = 0$. It is constructed from a single 9-$\overline{9}$ pair, with a tachyon field that is positive on one side and negative on the other. I would conjecture — but will not try to prove here — that the sign of the -1-brane amplitude is reversed in crossing this domain wall.

The sevenbrane is similarly constructed with two 9-$\overline{9}$ pairs and 2×2 real Γ matrices. I would conjecture, but will again not try to prove, that a zerobrane wavefunction picks up a factor of -1 under parallel transport about the sevenbrane.

These conjectures assert that there is a sort of discrete electric-magnetic duality between -1-branes and 8-branes, and between 0-branes and 7-branes. Recall that in ten dimensions, dual p-branes and q-branes carrying additive charges obey $p+q = 6$. In the case of branes carrying discrete charges, one apparently has $p + q = 7$.

A note on Bott periodicity. Finally, we make a note on Bott periodicity, which asserts that for KO-theory with compact support, one has $\mathrm{KO}(\mathbf{R}^n) = \mathrm{KO}(\mathbf{R}^{n+8})$. In particular, Bott periodicity maps the -1-brane to the 7-brane and the 0-brane to the 8-brane.

The periodicity map can be described as follows. Consider an element of $\mathrm{KO}(\mathbf{R}^n)$ described by trivial bundles (E_0, F_0) on $\mathbf{R}^n$ with a tachyon map $T_0 : F_0 \to E_0$. From this data one constructs an element of $\mathrm{KO}(\mathbf{R}^{n+8})$ by letting S_+ and S_- be the chiral spinor representations of $SO(8)$, and setting $E = E_0 \otimes (S_+ \oplus S_-)$, $F = F_0 \otimes (S_+ \oplus S_-)$. We also denote by $\vec{\Gamma} : S_- \to S_+$ the $SO(8)$ Gamma matrices, and by $\vec{x}$ the last eight coordinates of $\mathbf{R}^{n+8}$. Then in a hopefully evident notation one takes the tachyon

[15] Since every open string has two ends, the generator -1 of the center of $O(32)$ acts trivially on all open string states. The perturbative gauge group as opposed to the group acting on the Chan-Paton factors at the end of a string is thus $O(32)/\mathbf{Z}_2$. If one replaces $O(32)$ by $O(32)/\mathbf{Z}_2$, one gets an extra $\mathbf{Z}_2$ in π_1. This leads to the possibility of considering bundles without "vector structure," a generalization we will make in section 5.

field to be

$$T = \begin{pmatrix} T_0 & \vec{\Gamma} \cdot \vec{x} \\ \vec{\Gamma}^T \cdot \vec{x} & -T_0 \end{pmatrix}. \tag{4.20}$$

Comparing to (4.19) and (4.13), we see that the relation between the eight-brane and the zerobrane is a typical example of this periodicity map.

5. Some generalizations

In this section, we consider three types of generalization of the above discussion, involving orbifolds, orientifolds, and the incorporation of the Neveu-Schwarz three-form field H.

5.1 Orbifolds

The simplest case to consider is Type IIB superstring theory on an orbifold. For this, we begin with a spacetime manifold X, and seek to divide by a finite group G of symmetries of X. X is endowed with an orientation and spin structure, and these are preserved by G.

D-brane configurations on X/G are understood as G-invariant configurations of D-branes on X [4]. G in general may act in an arbitrary fashion on the gauge bundles supported on the D-branes. A D-brane configuration, as we have seen, represents in general a pair of bundles (E, F). This construction can be made in a completely G-invariant way (see, for example [15], section 2.3, for an introduction to such matters), so we can assume that G acts on (E, F). In tachyon condensation, we should assume that a pair of bundles (H, H) can be created or annihilated only if G acts on both copies of H in the same way. Otherwise, the requisite tachyon field would not be G-invariant.

Pairs of bundles (E, F) with G action, modulo the relation $(E, F) = (E \oplus H, F \oplus H)$ for any bundle H with G action, form a group called $\mathrm{K}_G(X)$. ($\mathrm{K}_G(X)$ is called the "G-equivariant K-theory of X". See again [15], section 2.3, for an introduction.) We conclude that for Type IIB superstrings on X/G, D-brane charge takes values in $\mathrm{K}_G(X)$.

For Type IIA, we similarly get $\mathrm{K}^1_G(X)$, and for Type I we get $\mathrm{KO}_G(X)$.

The standard string theory formula for the Euler characteristic of an orbifold X/G (in Type II string theory) has been shown [28] to coincide with the Euler characteristic in equivariant K-theory (understood as the dimension of $\mathrm{K}_G(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ minus that of $\mathrm{K}^1_G(X) \otimes_{\mathbf{Z}} \mathbf{Q}$). This is presumably related to the fact that the Betti numbers of the orbifold, in the string theory sense, determine the possible charges for Type IIB and Type IIA p-form fields, and those charges actually take values in $\mathrm{K}_G(X)$ or $\mathrm{K}^1_G(X)$, respectively.

5.2 Involutions

Now we specialize to the case that $G = \mathbf{Z}_2$, for which some additional constructions are possible. We denote the generator of $\mathbf{Z}_2$ as τ. Thus τ is a so-called "involution" of X, a symmetry with $\tau^2 = 1$.

Instead of simply dividing by the geometrical action of τ on X, we have three additional options:

1. We can divide by τ times Ω, the operator that reverses the orientation of a string.
2. We can divide by τ times $(-1)^{F_L}$, the operator that acts as -1 or $+1$ on states in a left-moving Ramond or Neveu-Schwarz sector.
3. We can divide by τ times the product $\Omega(-1)^{F_L}$.

More generally still, we could divide by a finite symmetry group G of spacetime which has some elements that act only geometrically and other elements that act also via Ω, $(-1)^{F_L}$, or $\Omega(-1)^{F_L}$. For brevity, I will not discuss this generalization, which combines the different cases.

I will briefly analyze the three types of $\mathbf{Z}_2$ action listed above. Since Ω acts on 9-brane (or $\overline{9}$-brane) bundles by complex conjugation, in case 1 we want to consider bundles (E, F) that are mapped by τ to their complex conjugates.[16] Whenever we say that a bundle, such as E, is mapped by τ to its complex conjugate $\overline{E}$, we mean, to be more precise, that $\tau^*(E)$ — the pullback of E by τ — is isomorphic to $\overline{E}$, and that an isomorphism $\psi : \tau^*(E) \to \overline{E}$ (obeying $(\psi\tau^*)^2 = 1$) is given.

Now we consider (E, F) to be equivalent to $(E \oplus H, F \oplus H)$, where H is similarly mapped by τ to its complex conjugate.

Such pairs make up a group that has been called $\mathrm{KR}(X)$ [29]. $\mathrm{KR}(X)$ depends, of course, on the choice of τ, but this is not usually indicated explicitly in the notation.

Now let us try to interpret case 2. Since $(-1)^{F_L}$ reverses the sign of D-brane charge, D-brane configurations on $X/\mathbf{Z}_2$ should in this case be related to D-brane configurations on X whose K-theory class is *odd* under $\mathbf{Z}_2$. This means that τ maps the pair (E, F) to (F, E). (This means that we are given isomorphisms $\lambda : (E, F) \to (\tau^*F, \tau^*E)$ with $(\lambda\tau^*)^2 = 1$.) We consider a trivial pair to be a pair (H, H) (with H isomorphic with τ^*H). The group of such pairs (E, F) with (E, F) equivalent to $(E \oplus H, F \oplus H)$ for any such H, make a K-like group that does not seem to have been much investigated mathematically. The name $\mathrm{K}_\pm(X)$ has been proposed for this group, and it has been argued by M. J. Hopkins that $\mathrm{K}_\pm(X)$ can be computed in terms of conventional equivariant K-theory as follows:

$$\mathrm{K}_\pm(X) = \mathrm{K}^1_{\mathbf{Z}/2}(X \times \mathbf{S}^1)\,. \tag{5.1}$$

[16] If a bundle E is defined with transition functions g_{ij} relative to an open cover U_i of X, then the complex conjugate of E is a bundle $\overline{E}$ with transition functions $\overline{g}_{ij}$.

Here $K_{\mathbf{Z}/2}$ is conventional equivariant cohomology for the group $G = \mathbf{Z}_2$; $\mathbf{Z}_2$ acts on $X \times \mathbf{S}^1$ by the product of the action of τ on X and an orientation-reversing symmetry of $\mathbf{S}^1$.

Examples in which τ acts together with $(-1)^{F_L}$ are interesting because only a few examples of stable nonsupersymmetric D-branes have been closely examined in the literature, and one of these [8, 11] is of this type. In those papers, an orbifold is considered in which space is $\mathbf{R}^9/\mathbf{Z}_2$, with $\mathbf{Z}_2$ acting by -1 on the last four coordinates of $\mathbf{R}^9$ and $+1$ on the first five, times $(-1)^{F_L}$. For this action of $\mathbf{Z}_2$ on $\mathbf{R}^9$, it has been shown by M. J. Hopkins (by using (5.1) plus the periodicity theorem) that (for $K_\pm$ with compact support) $K_\pm(\mathbf{R}^9) = \mathbf{Z}$. Thus, we expect a stable D-brane configuration carrying an additive conserved charge. This presumably is the configuration studied in [8, 11]. (In [8], it was described as a tachyonic soliton on a brane-antibrane pair, and in [11] as a D-brane.)

The final case is type 3, in which τ acts via $\Omega(-1)^{F_L}$ and hence maps (E, F) to $(\overline{F}, \overline{E})$. This combines KR theory with $K_\pm$. The D-branes charges live in a group that might be called $KR_\pm(X)$.

5.3 Incorporation of the B-field

So far in this paper, we have suppressed the role of the Neveu-Schwarz B-field. B has a three-form field strength H, and a characteristic class $[H] \in H^3(X, \mathbf{Z})$.

When $[H] \neq 0$, it is no longer true that Type IIB D-brane charge takes values in $K(X)$. Indeed, branes can be wrapped on a submanifold Z of spacetime only if ([13], eq. (3.13)) when restricted to Z

$$[H] + W_3(Z) = 0\,. \tag{5.2}$$

($W_3(Z)$ is of order two, and so can be placed on the left or right of this equation.) For $[H] = 0$, the condition is that $W_3(Z) = 0$; as we have seen in section 4, this is the right condition for K-theory. For $[H] \neq 0$, the condition is clearly no longer the right one for K-theory.

I will now argue that when $[H]$ is a torsion class (some examples of this type were studied in [13]), D-brane charge takes values in a certain twisted version of K-theory that will be described. I do not know the right description when $[H]$ is not torsion.

First recall the case with $[H] = 0$. We recall from section 4.3 that the "gauge bundle" on a D-brane is twisted in a subtle but important way. Cover X with open sets U_i, and describe $w_2(Z)$ by a $\{\pm 1\}$-valued cocycle ϕ_{ijk} on $U_i \cap U_j \cap U_k$. Then the gauge bundle of a D-brane is described by transition functions g_{ij} on $U_i \cap U_j$. The transition functions for a vector bundle would on $U_i \cap U_j \cap U_k$ obey $g_{ij} g_{jk} g_{ki} = 1$. Instead, in D-brane theory, the required condition is

$$g_{ij} g_{jk} g_{ki} = \phi_{ijk}\,. \tag{5.3}$$

(In a footnote in section 3, we proved that this condition implies that the ϕ_{ijk} obey the usual cocycle relation $\phi_{ijk}\phi_{ikl} = \phi_{jkl}\phi_{ijl}$ on quadruple overlaps.) For example, if $n = 1$, functions g_{ij} obeying (5.3) define a Spin_c structure on Z. This twisted condition was needed in section 4.3 to match to K-theory.

Since $[H]$ appears together with $W_3(Z)$ in the condition (5.2) for D-brane wrapping, one suspects that when $[H] \neq 0$, a cocycle defining $[H]$ should somehow be included in (5.3). I will describe how to do this when $[H]$ is torsion. Consider the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{i} \mathbf{R} \to U(1) \to 0\,, \tag{5.4}$$

with i the inclusion of $\mathbf{Z}$ in $\mathbf{R}$. The associated long exact sequence in cohomology reads in part

$$\cdots\ H^2(Z,\mathbf{R}) \xrightarrow{i} H^2(Z,U(1)) \to H^3(Z,\mathbf{Z}) \to H^3(Z,\mathbf{R}) \to \cdots \tag{5.5}$$

We conclude that $[H] \in H^3(Z,\mathbf{Z})$ maps to zero in $H^3(Z,\mathbf{R})$ — and so is a torsion class — if and only if $[H]$ can be lifted to an element in $H^2(Z,U(1))$, which we will call H^*.

The lift of $[H]$ to H^*, if it exists, is not necessarily unique. Exactness of (5.5) says that H^* is unique modulo addition of an element of the form $i(b)$, for any $b \in H^2(Z,\mathbf{R})$. Suppose that $[H]$ is of order n in $H^3(Z,\mathbf{Z})$. Then we can always pick its lift to $H^2(Z,U(1))$ so that H^* is of order n. We cannot make the order of H^* smaller than this, because $mH^* = 0$ implies $m[H] = 0$.

Being of order n, H^* can be represented by a cocycle valued in the n^{th} roots of unity. This means that on each $U_i \cap U_j \cap U_k$, we are given an n^{th} root of unity h_{ijk}, obeying the cocycle relation on quadruple overlaps.

Now, since $[H]$ appears together with $W_3(Z)$ in (5.2), I propose that the corresponding cocycles appear together in the generalization of (5.3). The "gauge bundle" on a D-brane would thus be described by transition functions obeying

$$g_{ij}g_{jk}g_{ki} = h_{ijk}\phi_{ijk}\,. \tag{5.6}$$

It should be possible to check this directly via worldsheet global anomalies.

Now let us specialize to the case of 9-branes (or $\overline{9}$-branes). In this case, we set $\phi_{ijk} = 1$, since X is spin. Hence, 9-brane gauge bundles are described by transition functions that obey

$$g_{ij}g_{jk}g_{ki} = h_{ijk}\,. \tag{5.7}$$

The direct sum of two such twisted bundles obeys obeys the same condition. So it is possible to define a twisted K-group $\mathrm{K}_{[H]}(X)$ whose elements are pairs (E,F) of such twisted bundles subject to the usual equivalence relation, which says that (E,F) is equivalent to $(E \oplus H, F \oplus H)$ for any H. In [30], it is shown that (5.6) is the condition for a submanifold W to determine a class in this kind of K-theory.

It is possible to describe twisted bundles more intrinsically, without talking about open covers and transition functions (which have been used here to try to keep things elementary). This approach, which is taken in the mathematical literature on $\mathrm{K}_{[H]}(E)$, proceeds as follows. If E is a twisted bundle, there are associated with it several ordinary bundles. There is a bundle $\mathbf{P}(E)$ of complex projective spaces. The obstruction to deriving $\mathbf{P}(E)$ by projectivizing a vector bundle is measured by the class $H^* \in H^2(Z, U(1))$. Also, the endomorphisms of E are valued in an ordinary vector bundle, whose sections make an algebra $A(E)$. In the mathematical literature, $\mathrm{K}_{[H]}(X)$ is defined in terms of modules for the algebra $A(E)$. One fundamental theorem [31] asserts that for any $[H]$ of finite order n, there exists a twisted bundle E of some finite rank m (which is always a multiple of n). Given this theorem, one proves as follows that $\mathrm{K}(X)$ and $\mathrm{K}_{[H]}(X)$ are equivalent rationally. Tensoring with E gives a map from $\mathrm{K}(X)$ to $\mathrm{K}_{[H]}(X)$; tensoring with E^* (the dual of E) gives a map back from $\mathrm{K}_{[H]}(X)$ to to $\mathrm{K}(X)$. The composite is multiplication by m^2 and so is an isomorphism rationally.

So far, our evidence that D-brane charge for Type IIB takes values in $\mathrm{K}_{[H]}(X)$ is mainly formal: $\mathrm{K}_{[H]}(X)$ is a natural modified version of $\mathrm{K}(X)$ that can be constructed from the data at hand, and extends (5.3) in a tempting way. The analogous statement for Type IIA is that D-brane charge is classifed by $\mathrm{K}^1_{[H]}(X)$; for Type I, it should be classified by $\mathrm{KO}_{[H]}(X)$.

We will now give strong support for this picture by showing that in the case of Type I, it is equivalent to something that is known independently. The worldsheet θ angles are odd under the projection that reduces Type IIB to Type I, so must take the values 0 or π in Type I. This implies that for Type I, $[H]$ is of order 2, and the cocycle h_{ijk} takes values in $\{\pm 1\}$. This means that H^* actually lies in the subgroup $H^2(X, \mathbf{Z}_2)$ of $H^2(X, U(1))$.

At $[H] = 0$, for Type I the 9-branes and $\overline{9}$-branes carry $SO(n)$ vector bundles. But if we turn on a non-zero $[H]$ which is of order 2, then (5.7) becomes the condition for an $SO(n)/\mathbf{Z}_2$ bundle without vector structure, in the sense of [32]. We recall that, just as the obstruction to spin structure of an $SO(N)$ bundle W is measured by a class $w_2(W) \in H^2(X, \mathbf{Z}_2)$, so the obstruction to vector structure for an $SO(n)/\mathbf{Z}_2$ bundle V is measured by a class $\widetilde{w}_2(V) \in H^2(X, \mathbf{Z}_2)$. $\widetilde{w}_2(V)$ can be represented by a $\{\pm 1\}$-valued cocycle t_{ijk} on $U_i \cap U_j \cap U_k$. An $SO(n)/Z_2$ bundle whose vector structure is obstructed by a cocycle t has transition functions g_{ij} that obey $g_{ij}g_{jk}g_{ki} = t_{ijk}$. (The cocycle would cancel out if we take the matrices g_{ij} in the adjoint representation of $SO(n)$, or any other representation in which the central element -1 of $SO(n)$ acts trivially.) So (5.7) asserts that the cohomology class H^* of the B-field equals $\widetilde{w}_2(V)$. This statement is true [33]. Therefore, Type I D-brane charge takes values in $\mathrm{KO}_{[H]}(X)$, giving strong encouragement to the expectation that the analogous statements are true for Type II.

JHEP12(1998)019

Comparison to cohomology theory. One point that now requires some discussion is why we can get this kind of description only if $[H]$ is torsion.

K-theory is regarded as a generalized cohomology theory (see for example section 2.4 of [15]). To get some intuition about K-theory with nonzero $[H]$, we might consider a hierarchy in which at level one one has cohomology theory and gauge fields, and at level two one has K-theory and two-form B-fields. (Level three might consist of elliptic cohomology and some stringy construction, but that remains to be seen.)

There is a notion of cohomology theory coupled to any gauge field A with zero curvature (cohomology with values in any flat bundle; the analogy we are about to make is more precise if the bundle is a line bundle). The level two analog should be K-theory coupled to any flat B-field. A gauge field whose curvature is zero is one whose Chern classes are torsion. Similarly, a flat B-field is one whose characteristic class $[H]$ is torsion. A candidate for K-theory coupled to a flat B-field is our friend $\mathrm{K}_{[H]}(X)$.

What if $[H]$ is not torsion? Let us compare to what happens for cohomology. One can couple differential forms to any vector bundle with connection A, replacing the usual exterior derivative by its gauge-covariant extension $d_A = d + A$. But if the curvature of A is not zero, one no longer gets a cohomology theory, since $d_A^2 \neq 0$. By analogy, one cannot expect to define a generalized cohomology theory when $[H]$ is not torsion; one must expect to go "off shell" in some way. There is no obvious known mathematical theory; Type II string theory itself may be the only candidate. Perhaps there is an approach via noncommutative geometry; so far, noncommutative geometry has been used to describe D-branes coupled to flat but irrational and topologically trivial B-fields [34].

Orbifolds with discrete torsion. D-branes on an orbifold with discrete torsion will lead, in view of the analysis in [35], to a mixture of two of the constructions that we have considered. In this case, instead of the K-theory of pairs (E, F) of bundles with G action, one wants the K-theory of such pairs with a projective action of G (with a fixed cocycle determined by the discrete torsion). This presumably should be understood as a G-equivariant version of $\mathrm{K}_{[H]}$.

6. Stringy constructions for Type I

In this section, we will discuss the worldsheet construction of the zerobrane and -1-brane of Type I. Actually, up to a certain point the discussion can be carried out equally well for Type I or Type IIB. However, since the generators of $\mathrm{KO}(\mathbf{S}^9)$ and $\mathrm{KO}(\mathbf{S}^{10})$ are mapped to zero if one forgets the reality condition of the bundles and maps KO-theory to ordinary K-theory, we expect that the stable zerobrane and -1-brane of Type I correspond to objects that are unstable if considered in Type IIB. Thus in the Type IIB description, we expect to see a tachyon that is removed by the orientifold projection.

JHEP12(1998)019

6.1 The zerobrane

We consider first the zerobrane, which has already been analyzed [10]; we will aim to clarify a few points. (The discussion applies equally well to the eightbrane, as we briefly note later.) From the point of view of K-theory, the Type I zerobrane is described by the same tachyon field $T = \vec{\Gamma} \cdot \vec{x}$ as the supersymmetric branes. This suggests that we should interpret the zerobrane as a D-brane, much like the more familiar supersymmetric branes.

The naive idea is then to introduce in Type I a D-particle — located, say, at $x^1 = \ldots = x^9 = 0$. One immediately runs into the following oddity (which corresponds to the factor of $\sqrt{2}$ in the multiplicity of states noted at the end of section 4). Type I superstring theory also has 9-branes, so there are 0-9 open strings that must be quantized. In the Neveu-Schwarz 0-9 sector, the fermions $\psi_1, \ldots, \psi_9$ (superpartners of $x^1, \ldots, x^9$) have zero modes which we may call $w_1, \ldots, w_9$. We therefore have to quantize an *odd*-dimensional Clifford algebra,

$$\{w_i, w_j\} = 2\delta_{ij}, \qquad i, j = 1, \ldots, 9\,. \tag{6.1}$$

There is, however, no satisfactory quantization of such an odd-dimensional Clifford algebra.[17] The nine-dimensional Clifford algebra has two irreducible representations, each of dimension 16, and differing simply by $w_i \to -w_i$. (The two representations of the Clifford algebra are equivalent as representations of Spin(9).) In one representation, the product $\overline{w} = w_1 w_2 \cdot \ldots \cdot w_9$, which is in the center of the Clifford algebra, is represented by $+1$, and in the other representation, $\overline{w} = -1$. Generally, in quantum field theory, we should use an irreducible representation of the algebra of observables. In this case, we have the problem that in an irreducible representation, there is no operator $(-1)^F$ that anticommutes with the w_i (such an operator would clearly change the sign of $\overline{w}$ and so interchange the two representations of the Clifford algebra). Without a $(-1)^F$ operator, we cannot make sense of the worldsheet sum over spin structures. To have a $(-1)^F$ operator, we must include both representations of the Clifford algebra (and let $(-1)^F$ exchange them). But what is a natural explanation of this doubling of the worldsheet spectrum?

To account for this doubling, we should have a tenth fermion zero mode, say η, on the 0-9 string. Then the doubling of the spectrum arises because the (unique) irreducible representation of the ten-dimensional Clifford algebra is 32-dimensional, and decomposes under $w_1, \ldots, w_9$ as the direct sum of the two 16-dimensional representations of the nine-dimensional Clifford algebra. The operator that anticommutes with $\overline{w}$ is η, and the operator that acts as $(-1)^F$ on the zero modes (anticommuting with η as well as the w's) is $\eta\overline{w}$. Making the GSO projection has the effect, on the string ground state, of reducing from 32 states to a single irreducible 16-dimensional

[17] In the Ramond 0-9 sector, one gets the same basic problem of an odd-dimensional Clifford algebra — in this case $\psi_1, \ldots, \psi_9$ have no zero mode but ψ_0 does.

representation of the smaller Clifford algebra. Thus, for the sake of counting states, the spectrum is the same as if we use an irreducible representation of the original nine-dimensional Clifford algebra and do not make a GSO projection. But adding the extra fermion zero mode and making the GSO projection gives a way to get this spectrum that is more coherent with the rest of string theory.

To obtain this extra fermion zero mode for 0-9 strings, we postulate that on any boundary of an open string that lies on the zerobrane, there is a field $\eta(\tau)$ (τ being a parameter along the boundary) with Lagrangian

$$L = i \int d\tau \; \eta \frac{d\eta}{d\tau}. \tag{6.2}$$

Quantization of this Lagrangian gives the required η zero mode.

Now we can describe some rules for worldsheet computations. Consider a worldsheet Σ with a boundary component S (which is a circle, of course) on the zerobrane. The spin structure of Σ when restricted to S may be in either the Neveu-Schwarz or Ramond sector. In the NS sector, η is an antiperiodic function on S, and in the Ramond sector, it is periodic. We assume that the worldsheet path integral includes an integral over η which (if any vertex operators on S are independent of η) is simply

$$\int D\eta(\tau) \exp\left(i \oint_S d\tau \eta \frac{d\eta}{d\tau} \right). \tag{6.3}$$

This integral is easy to calculate. It equals 0 in the Ramond sector, and $\sqrt{2}$ in the Neveu-Schwarz sector. The vanishing in the Ramond sector arises because there is an η zero mode in the path integral — the constant mode of $\eta(\tau)$. As for the NS path integral, as the Hamiltonian is zero, one would expect it to count the number of states obtained by quantizing the field η. We cannot quite give this path integral that interpretation, because this system has no natural quantization; trying to quantize it, we get a one-dimensional Clifford algebra (generated by η), which like any odd-dimensional Clifford algebra has no natural quantization. However, if we had *two* η fields, the path integral

$$\int D\eta_1(\tau) \, D\eta_2(\tau) \;\; \exp\left(i \sum_{i=1}^{2} \oint_S \eta_i \frac{d\eta_i}{d\tau} \right) \tag{6.4}$$

would equal 2, because in this case the quantum system (a two-dimensional Clifford algebra) can be naturally quantized and has two states. So for one η field, the path integral equals $\sqrt{2}$. All the usual factors in the worldsheet path integral must be supplemented with this factor, which was found in [10] from another point of view. Because of this factor, a zerobrane of this kind in Type IIB has a mass greater than the mass of a conventional Type IIA zerobrane (with the same values of α' and the string coupling) by a factor of $\sqrt{2}$.

The vanishing of the η path integral in the Ramond sector assumes that the 0-0 vertex operators, inserted on the zerobrane boundary, are independent of η. What in fact do those operators look like? Since $d\eta/d\tau = 0$ by the η equation of motion, and $\eta^2 = 0$ by fermi statistics, the possible vertex operators are at most linear in η, and hence take the form $\mathcal{O}(X,\psi)$ or $\eta\mathcal{O}'(X,\psi)$. (X and ψ are the worldsheet matter fields, and for brevity we omit ghosts from the notation.) A simple generalization of the reasoning by which we analyzed the path integral (6.3) shows that an amplitude with an odd number of vertex operator insertions of the form $\eta\mathcal{O}'$ on S receives a contribution only from the Ramond sector (that is, from spin structures that restrict on S to the Ramond sector), while an amplitude with an even number of such insertions receives a contribution only from the NS sector.

Note that η is of conformal dimension zero, so for $\eta\mathcal{O}'$ to be of dimension one, $\mathcal{O}'$ is of dimension one. The GSO projection as usual projects out the NS tachyon of type $\mathcal{O}$. But since η is odd under $(-1)^F$, the states with vertex operators of type $\eta\mathcal{O}'(X,\psi)$ undergo the opposite of the usual GSO projection. The $\eta\mathcal{O}'$ tachyon therefore survives the GSO projection.

Thus, this zerobrane, regarded as an excitation of Type IIB superstring theory, is tachyonic. To understand what happens for Type I, one must still analyze the Ω projection, which reverses the orientation of the open string. The zerobrane will be stable in Type I if the Ω projection removes the tachyon from the 0-0 sector.

That it does so has been deduced by Sen from another construction (in which the zerobrane is built by a marginal deformation of a 1-$\overline{1}$ system [10]). Sen also suggested the following direct approach to the question. Since we know how the Ω operator acts on closed strings, we can deduce how it acts on open strings by looking at transitions between 0-0 open strings and closed strings. The simplest worldsheet describing such a transition is a disc D, with boundary ending on the zerobrane. We consider an amplitude with one 0-0 vertex operator on the boundary of the disc, and one closed string operator in the interior. We consider the case that the states making the transition are bosonic. We want to look at a two-point function of the form $\langle\mathcal{W}\cdot\eta\mathcal{O}'\rangle$, with $\mathcal{W}$ a closed string vertex operator in the interior of the disc, and $\eta\mathcal{O}'$ an open string vertex operator.

As we have discussed before, since there are an odd number of $\eta\mathcal{O}'$ insertions, such an amplitude can be nonzero only if the spin structure on the boundary of D is in the Ramond sector. This is possible only if the vertex operator $\mathcal{W}$ is a Ramond-Ramond vertex operator, which creates a "cut" in the worldsheet fermions. In fact, we take $\mathcal{W}$ to be the vertex operator of the massless RR scalar of Type IIB. We can write this in the $(-1/2,-1/2)$ picture as $\mathcal{W} = e^{-(\phi(z)+\widetilde{\phi}(\overline{z}))/2}k\cdot\Gamma_{\alpha\beta}S^{\alpha}(z)\widetilde{S}^{\beta}(\overline{z})e^{ik\cdot X}$ with z a complex parameter on the disc, k the momentum, and S and $\widetilde{S}$ the left and right-moving spin fields. This particular state is odd under Ω and is projected out in reducing to Type I. So we can show that the 0-0 tachyon is odd under Ω by showing that it can make a transition to the RR scalar.

The vertex operator of the 0-0 tachyon, in the -1 picture, is $\eta\mathcal{O}'(\tau) = \eta e^{-\phi}e^{iq\cdot X}$. ($q$ points in the "time" direction, since the 0-0 tachyon propagates only on the zerobrane.) The matrix element $\langle \mathcal{W}\cdot\eta\mathcal{O}'\rangle$ is nonzero, since η gets an expectation value due to the zero mode, while nonvanishing of the matter and ghost matrix element is equivalent to the statement [1] that Type IIA zerobranes carry RR charge.

One could in exactly the same way construct a Type I eightbrane; η is included in the same way. Suppose that we want instead a Type I p-brane for $p = 2, 4$, or 6. For $p = 2, 6$, we run into the following. After adding the η field, we have $p + 2$ fermion zero modes in the NS sector of the p-9 string. This is an even number, which enables us to make sense of the GSO projection. But after imposing the GSO projection, we are left with the chiral spinors of $SO(1, p+1)$ (obtained by quantizing $p+2$ fermion zero modes, one of which has "timelike" metric). These chiral spinors are complex, contradicting the fact that the p-9 wavefunctions should be real. So there is apparently no twobrane or sixbrane. Likewise, for a fourbrane, we would meet chiral spinors of $SO(1, 5)$, which are pseudoreal, rather than real. So there is no fourbrane either.[18] These results agree with $\mathrm{KO}(\mathbf{S}^3) = \mathrm{KO}(\mathbf{S}^5) = \mathrm{KO}(\mathbf{S}^7) = 0$, $\mathrm{KO}(\mathbf{S}^1) = \mathrm{KO}(\mathbf{S}^9) = \mathbf{Z}_2$.

Suppose, however, that the ninebranes were quantized with symplectic rather than orthogonal Chan-Paton factors. (There is no supersymmetric way to do this with tadpole cancellation, but up to a certain point we can consider such a theory anyway.) D-brane charge would then take values in a K-group $\mathrm{KSp}(X)$ whose elements are pairs (E, F) of symplectic bundles, modulo the usual sort of equivalence relation. The symplectic or pseudoreal Chan-Paton factors of the ninebrane would cancel the reality problem for the fourbrane while creating one for the zerobrane and eightbrane. So this kind of theory has a fourbrane but no other even-dimensional branes. This is in agreement with the results of Bott periodicity, according to which $\mathrm{KSp}(\mathbf{S}^5) = \mathbf{Z}_2$, $\mathrm{KSp}(\mathbf{S}^1) = \mathrm{KSp}(\mathbf{S}^3) = \mathrm{KSp}(\mathbf{S}^7) = \mathrm{KSp}(\mathbf{S}^9) = 0$.

Comparison to Gimon-Polchinski. One might wonder whether we instead could use the arguments of Gimon and Polchinski [36] to learn how Ω acts on the 0-0 strings. Their approach would entail examining the operator product of 0-9 and 9-0 vertex operators. These vertex operators require some novelty, since spin fields for an odd number of fermions are not usually considered in conformal field theory, and the part of the 0-9 vertex operator involving η is also somewhat unusual. I will not try to make this analysis here.

Spinor quantum numbers and multiplicative conservation law. To tie up the discussion with what was said in section 2, we note finally that as explained in [10], the Type I zerobrane transforms in the spinor representation of $SO(32)$. In

[18] If we include symplectic fourbrane Chan-Paton factors to make the 4-9 strings real, we find that the 4-4 spectrum has a tachyon.

fact, in the Ramond 0-9 sector, there are two fermion zero modes (modes of η and ψ^0), whose quantization gives two states, of which one obeys the GSO projection. Allowing for the 9-brane Chan-Paton factors, this gives a single $SO(32)$ vector of fermion zero modes, whose quantization gives a spinor representation of $SO(32)$. (There also are fermion zero modes, coming from other sectors, that are $SO(32)$-invariant.)

Suppose we are given a set of k coincident Type I zerobranes. Then the tachyon vertex operators depend on a $k \times k$ matrix M which acts on the Chan-Paton factors, and takes the form $\mathcal{V}(M) = M \cdot \mathcal{V}_0$, where $\mathcal{V}_0$ is the tachyon vertex operator as analyzed above. The Ω projection, in addition to the action found above, maps $M \to M^T$. If M is antisymmetric, it is odd under Ω, so the $\eta\mathcal{O}'$ tachyons with antisymmetric M survive the Ω projection. An antisymmetric M has an even number of nonzero eigenvalues. Every pair of nonzero eigenvalues describes (presumably) the flow toward annihilation of a pair of zerobranes. So the zerobrane number is conserved only modulo 2, as expected.

6.2 The -1-brane

We have seen in section 4.5 that the Type I -1-brane is understood in K-theory as a Type IIB -1 brane-antibrane pair that are exchanged by complex conjugation. So, in a worldsheet construction, we will try to understand this object by starting in Type IIB with a -1-brane and antibrane, and assuming that they are exchanged by worldsheet orientation reversal Ω.

Somewhat more generally, consider in Type IIB a system consisting of a coincident p-brane and $\overline{p}$-brane, for $p = -1, 3$, or 7. These values are selected because they are the values for which Ω reverses the sign of RR charge and maps Type IIB p-branes to $\overline{p}$-branes. As we reviewed in section 3, the p-$\overline{p}$ system in Type IIB has a tachyon which arises because the GSO projection for p-$\overline{p}$ open strings is opposite to the usual GSO projection. The p-$\overline{p}$ system hence exists for Type IIB but is unstable.

The only hope of stabilizing it for Type I is that the Ω projection might remove the tachyon from the p-$\overline{p}$ system. Note that there would be no hope of this for $p = 1, 5$, or 9 — the values for which RR charge is Ω-invariant. In these cases, Ω maps p-branes to p-branes and $\overline{p}$ to $\overline{p}$; so it maps p-$\overline{p}$ open strings to $\overline{p}$-p open strings. Ω hence cannot eliminate the open string tachyon for these values of p; it merely relates the p-$\overline{p}$ tachyon to the $\overline{p}$-p tachyon.

Instead, for $p = -1, 3, 7$, Ω exchanges p-branes with $\overline{p}$-branes. Hence, for those values of p, Ω maps p-$\overline{p}$ open strings to themselves (as a result of exchanging the two ends of the string but also turning p into $\overline{p}$ and vice-versa), and likewise maps $\overline{p}$-p open strings to themselves. Hence it is conceivable that for $p = -1, 3$, or 7, the Ω projection might remove the tachyon and stabilize the p-$\overline{p}$ system.

For $p = 1, 5$, or 9, Ω maps p-p open strings to themselves and hence, if one considers N parallel p-branes, Ω reduces the gauge group from $U(N)$ to $SO(N)$ or $Sp(N)$. Which reduction is made depends on how Ω acts. Gimon and Polchinski [36] gave a systematic procedure for showing that one gets $SO(N)$ for $p = 1$ or 9 and $Sp(N)$ for $p = 5$.

For $p = -1, 3$ or 7, there is no analogous question of reduction of the gauge group. Given a system of N parallel p-brane-antibrane pairs, Ω maps the p-p open strings to $\overline{p}$-$\overline{p}$ open strings and hence identifies the $U(N)$ of the p-p sector with the $U(N)$ of the $\overline{p}$-$\overline{p}$ sector; the unbroken gauge group is a diagonal $U(N)$ regardless of any phases in the action of Ω.

In summary then:

1. For $p = 1, 5$, or 9, one must analyze the Ω action on the p-p sector to understand what kind of gauge group the branes carry, but there is no sharp question about the p-$\overline{p}$ sector, which is simply mapped to $\overline{p}$-p.

2. For $p = -1, 3$, or 7, one must analyze the Ω action on the p-$\overline{p}$ sector to determine whether the p-$\overline{p}$ configuration is stable in Type I, but there is no sharp question about the p-p sector, which is simply mapped to $\overline{p}$-$\overline{p}$.

Though the questions of interest are thus rather different for $p = -1$, 3, or 7 than what they are for $p = 1, 5$, or 9, they can be answered in the same way, using arguments by Gimon and Polchinski [36]. The basic idea is to let $\mathcal{V}$ be a physical p-9 vertex operator. Denote as $\mathcal{V}^\Omega$ the transform of $\mathcal{V}$ by Ω; it is a 9-p or 9-$\overline{p}$ vertex operator depending on whether p is congruent to 1 or -1 modulo 4. A vertex operator that arises as a pole in the $\mathcal{V} \cdot \mathcal{V}^\Omega$ operator product would be a physical p-p or p-$\overline{p}$ vertex operator; by identifying it, we can learn which p-p or p-$\overline{p}$ vertex operators survive the Ω projection.

In the p-9 NS sector, there are $9-p$ fermion zero modes. In fact, the worldsheet fermions, which we label as ψ_i, $i = 0, \ldots, 9$ (with the p-brane spanning the first $p+1$ coordinates) have zero modes precisely if $i > p$. The NS ground state thus transforms in a spinor representation of $SO(9-p)$. (For our present purposes, the number $9-p$ is even, an important fact, as we have seen!) After imposing the GSO projection, the 9-p ground states actually transform as an $SO(9-p)$ spinor of definite chirality, say positive. The vertex operator of such a state, in the -1 picture, is of the form $\mathcal{V}(\epsilon) = e^{-\phi}\epsilon^\alpha S_\alpha e^{ik\cdot X}$, where S_α are the positive chirality spin fields and ϵ is a c-number wave-function in the spinor representation.

It is convenient to divide the worldsheet fermions in the p-$\overline{p}$ sector as ψ'_i, $i = 0, \ldots, p$ and $\psi''_j, j > p$. Neither ψ'_i nor ψ''_j has a zero mode in the NS sector. Consider a p-$\overline{p}$ vertex operator of the general kind $\mathcal{O}(X, \psi', \psi'')$. The vertex operators also may contain derivatives of the fields with respect to τ (a parameter on the boundary of the worldsheet), but this is not shown in the notation. To determine how a vertex operator transforms under Ω, one includes a factor of -1 for each τ derivative, a factor of $-i$ for each ψ', and a factor of $+i$ for each ψ''. (Note here that ψ' and ψ'' obey opposite boundary conditions on the boundary of the worldsheet and so transform under Ω with opposite phases. The phases are $\pm i$, not ± 1, because $\Omega^2 = (-1)^F$ for open strings, and ψ', ψ'' are both odd under $(-1)^F$.) In addition to these factors, Ω acting on the p-$\overline{p}$ sector gives a fixed additional phase α, coming from its action on the Chan-Paton wavefunction. This phase can be determined by looking at the Ω action on $\mathcal{V}(\epsilon) \cdot \mathcal{V}(\epsilon)^\Omega$; α must be such that that state, which we know must be present in the spectrum, survives the Ω projection.

In particular, for $p = -1, 3$, or 7, we want to determine whether $\mathcal{V}(\epsilon) \cdot \mathcal{V}(\epsilon)^\Omega$ transforms the same way as the p-$\overline{p}$ tachyon or oppositely to it. This will determine whether the tachyon is present in the spectrum or not. Note that the vertex operator for the tachyon is in the zero picture $\mathcal{W} = k \cdot \psi' \, e^{ik\cdot X}$. It transforms under Ω as $-i\alpha$.

As explained by Gimon and Polchinski, $\mathcal{V}(\epsilon) \cdot \mathcal{V}(\epsilon)^\Omega$ transforms under Ω as $i^{(9-p)/2}\alpha$. The factor of $i^{(9-p)/2}$ arises because if ϵ is a highest weight state, then $V(\epsilon) \cdot V(\epsilon)^\Omega \sim (\psi'')^{(9-p)/2}$. Since this state must survive in the spectrum, one has $\alpha = -i^{(9-p)/2}$. Hence the p-$\overline{p}$ tachyon transforms as $i^{1+(9-p)/2}$. It is therefore projected out for $p = -1$ or 7, but survives for $p = 3$. Hence, Type I has a stable -1-brane and a stable sevenbrane, but no stable threebrane. This result reflects the facts $\mathrm{KO}(\mathbf{S}^2) = \mathrm{KO}(\mathbf{S}^{10}) = \mathbf{Z}_2$, $\mathrm{KO}(\mathbf{S}^6) = 0$.

It is interesting to note that this result would be reversed if the ninebranes are quantized with symplectic rather than orthogonal Chan-Paton factors. Then $\mathcal{V}(\epsilon)$ carries a symplectic Chan-Paton label which gives an extra minus sign in the transformation of $\mathcal{V} \cdot \mathcal{V}^\Omega$.[19] As a result, in such a theory, there is a stable threebrane but no stable -1-brane or sevenbrane. This result reflects the facts that $\mathrm{KSp}(\mathbf{S}^6) = \mathbf{Z}_2$, $\mathrm{KSp}(\mathbf{S}^2) = \mathrm{KSp}(\mathbf{S}^{10}) = 0$, so in such a theory, one expects a threebrane but no -1-brane or sevenbrane.

Finally, note that Gimon and Polchinski prove [36] using the above factor of $i^{(9-p)/2}$ that for a system of only p-branes with $p = -1, 3$, or 7, one cannot define an Ω projection. This argument uses the fact that for p-branes only, $\Omega(\Omega^T)^{-1}$ should act on the Chan-Paton labels as a c-number. If one has both p-branes and $\overline{p}$-branes, then $(-1)^F$ acts nontrivially on the Chan-Paton labels, so the Ω action is more involved, and the problem found in [36] does not arise.

Symmetry breaking and $\mathbf{Z}_2$ conservation law. To compare to what was said in section 2, let us now show that the amplitude due to a Type I -1-brane is odd under

[19] If one endows $\mathcal{V}$ with a symplectic index and tries to find in $\mathcal{V} \cdot \mathcal{V}^\Omega$ the same p-$\overline{p}$ operator as before, one instead gets zero because of antisymmetry in the contraction of the symplectic indices of the operators. To get a nonzero result, one may, for example, look in the operator product $\mathcal{V}(-\tau)\mathcal{V}^\Omega(\tau)$ for an operator that appears with a coefficient odd under $\tau \to -\tau$; this oddness corresponds to an extra factor of -1 in the transformation under Ω.

the disconnected component of $O(32)$. As in section 2, this happens because in the presence of a -1-brane, there is a single $SO(32)$ vector of fermion zero modes (plus other zero modes that are $SO(32)$-invariant). They arise from the -1-9 Ramond strings. In quantizing this sector, there are no worldsheet fermion zero modes, so there is a unique ground state for each value of the Chan-Paton labels; by worldsheet supersymmetry its energy is zero. For one of the two possible -1-brane labels, this state survives the GSO projection; it also transforms as a vector of $SO(32)$ because of the 9-brane Chan-Paton labels, and this gives the expected multiplet of zero modes.

That the number of such -1-branes is conserved only modulo 2 may be seen, as for zerobranes, by observing that in a system of k coincident Type I -1-branes, there is a tachyon described by a $k \times k$ antisymmetric matrix M.

Acknowledgments

This work was supported in part by NSF Grant PHY-9513835. I would like to thank P. Deligne, D. Freed, M. J. Hopkins, S. Martin, G. B. Segal, and A. Sen for many helpful explanations.

References

[1] J. Polchinski, *Dirichlet branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 hep-th/9510017].

[2] M. Bershadsky, C. Vafa and V. Sadov, *D-Branes And Topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9511222].

[3] G. Moore and R. Minasian, *K-theory and Ramond-Ramond charges*, *J. High Energy Phys.* **11** (1997) 002 [hep-th/9710230].

[4] M.R. Douglas and G. Moore, *D-branes, quivers, and ALE instantons,* hep-th/9603167.

[5] P. Horava, *Equivariant topological sigma models*, *Nucl. Phys.* **B 418** (1994) 571 [hep-th/9309124.

[6] A. Sen, *Stable non-BPS states in string theory,* *J. High Energy Phys.* **06** (1998) 007 [hep-th/9803194].

[7] A. Sen, *Type I D-particle and its interactions*, *J. High Energy Phys.* **10** (1998) 021 [hep-th/9809111].

[8] A. Sen, *Stable non-BPS bound states of BPS D-branes*, *J. High Energy Phys.* **08** (1998) 010 [hep-th/9805019].

[9] A. Sen, *Tachyon condensation on the brane antibrane system*, *J. High Energy Phys.* **08** (1998) 012 [hep-th/9805170].

JHEP12(1998)019

[10] A. Sen, $SO(32)$ *Spinors of type I and other solitons on brane-antibrane pair,* *J. High Energy Phys.* **09** (1998) 023 [hep-th/9808141].

[11] O. Bergman and M.R. Gaberdiel, *Stable non-BPS D-particles,* *Phys. Lett.* **B 441** (1998) 133 [hep-th/9806155].

[12] M. Srednicki, *IIB or not IIB,* *J. High Energy Phys.* **08** (1998) 005 [hep-th/9807138].

[13] E. Witten, *Baryons and branes in anti-de Sitter space,* *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[14] D. Freed and E. Witten, to appear.

[15] M.F. Atiyah, *K-theory* W.A. Benjamin, New York 1967, reprinted with other useful references in *Michael Atiyah collected works*, Vol. 2, *K-theory* Clarendon Press, Oxford 1988.

[16] R. Bott, *Lectures on* $\mathrm{K}(X)$ W. A. Benjamin, New York, 1969.

[17] M.J. Duff and J.X. Lu, *Strings from fivebranes,* *Phys. Rev. Lett.* **66** (1991) 1402;
J.A. Dixon, M.J. Duff and J.C. Plefka, *Putting string/five-brane duality to the test,* *Phys. Rev. Lett.* **69** (1992) 3009, [hep-th/9208055].

[18] R. Jackiw and C. Rebbi, *Spin from isospin in a gauge theory,* *Phys. Rev. Lett.* **36** (1976) 1116.

[19] M.B. Green, *Point-like states for Type IIB superstrings,* *Phys. Lett.* **B 329** (1994) 435 [hep-th/9403040].

[20] V. Periwal, *Antibranes and crossing symmetry,* hep-th/9612215.

[21] E. Gava, K.S. Narain and M. H. Sarmadi, *On the bound states of* p- *and* $(p+2)$*-branes,* *Nucl. Phys.* **B 504** (1997) 214 [hep-th/9704006].

[22] C.G. Callan and J.M. Maldacena, *Brane dynamics from the Born-Infeld action,* *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147].

[23] K.G. Savvidy, *Brane death via Born-Infeld string,* hep-th/9810163.

[24] D. Quillen, *Superconnections and the Chern character,* *Topology* **24** (1985) 89.

[25] M.R. Douglas, *Branes within branes,* hep-th/9512077.

[26] M.F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, *Topology* **3** (1964) 3, reprinted in *Michael Atiyah collected works*, op. cit.

[27] E. Witten, *Small instantons in string theory,* *Nucl. Phys.* **B 460** (1996) 541 [hep-th/9511030].

[28] M.F. Atiyah and G.B. Segal, *On equivariant Euler characteristics,* *J. Geom. Phys.* **6** (1989) 671.

JHEP12(1998)019

[29] M.F. Atiyah, *K-theory and reality, Quart. J. Math. Oxford* (2) **17** (1966) 367, reprinted in *Michael Atiyah collected works*, op. cit.

[30] P. Donovan and M. Karoubi, *Graded Brauer groups and K-theory with local coefficients, IHES Pub.* **38** (1970) 5.

[31] A. Grothendieck, *Le groupe de Brauer,* section 1.6, in *Dix exposés sur la cohomologie des schémas*, North Holland 1968.

[32] M. Berkooz, R.G. Leigh, J. Polchinski, J.H. Schwarz, N. Seiberg and E. Witten, *Anomalies, dualities and topology of $D = 6$ $N = 1$ superstring vacua, Nucl. Phys.* **B 475** (1996) 115 [hep-th/9605184].

[33] A. Sen and S. Sethi, *The mirror transform of Type I vacua in six dimensions, Nucl. Phys.* **B 499** (1997) 45 [hep-th/9703157].

[34] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori, J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[35] M. Douglas, *D-branes and discrete torsion,* hep-th/9807235.

[36] E. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds, Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

JHEP12(1998)019